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ABSTRACT

This working paper is a product of a graduate research seminar in mathematics education at the University of Wisconsin. Chapter I contains the rationale for re-examining fractions. The discussion centers around the different interpretations of fractions in view of the use of hand-held calculators and the metric system. In Chapter II a logical, conceptual hierarchy of initial fraction concepts is presented. Chapter III is an extensive review of the related research. This review includes overviews of the research, dissertations, large-scale investigations, foundation work, further studies on readiness, and interpretations and recommendations. Chapter IV is a brief review of how fractions are presented in selected textbooks. Five pilot studies are described in Chapter V. Chapter VI includes an overview of the Developing Mathematical Processes (DMP) approach to fractions. (MK)

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Working Paper No. 218
Part 1 of 2 Parts

THE TEACHING OF INITIAL FRACTIONS CONCEPTS

by Maurice Burke, Nancy Hiber,
and Thomas Romberg

U.S. DEPARTMENT OF HEALTH,
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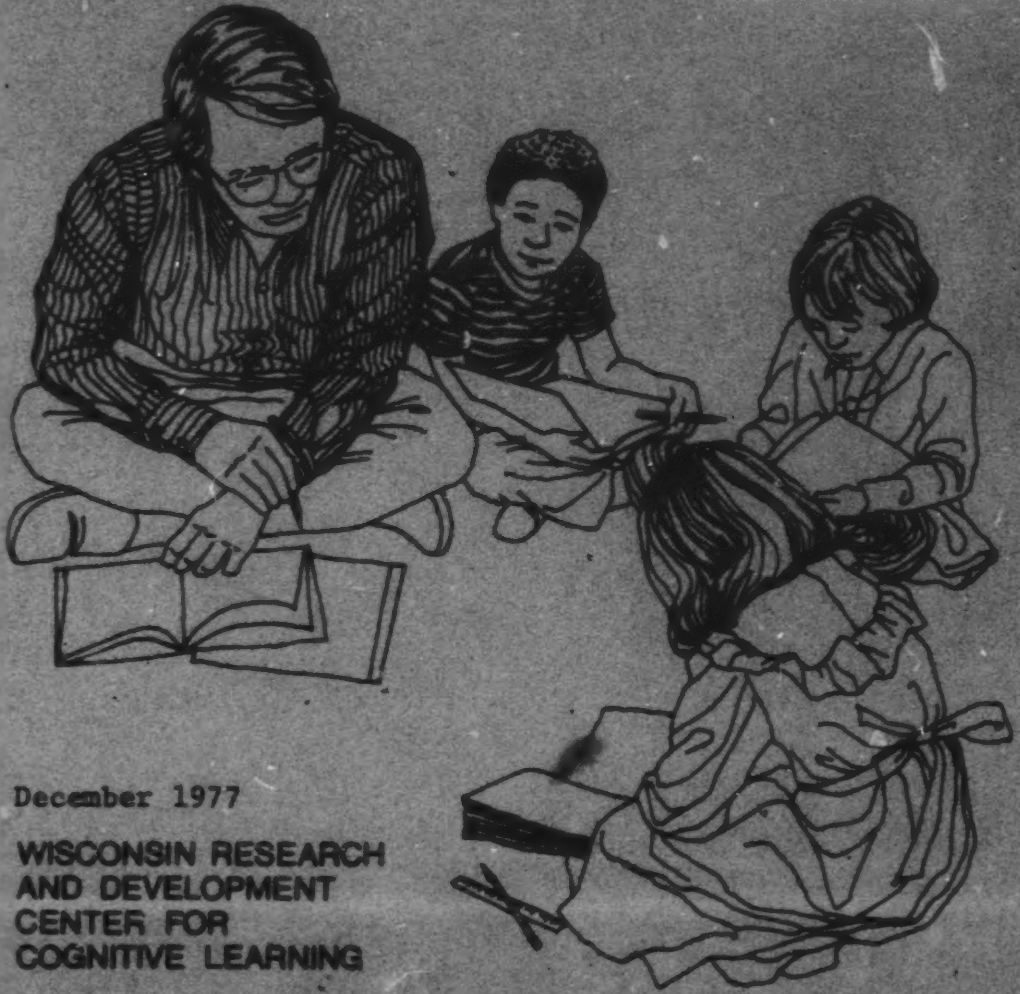

December 1977

**WISCONSIN RESEARCH
AND DEVELOPMENT
CENTER FOR
COGNITIVE LEARNING**

SE 029 216

Working Paper No. 218
Part 1 of 2 Parts

THE TEACHING OF INITIAL FRACTIONS CONCEPTS

by

Maurice Burke, Nancy Hiber, and Thomas Romberg

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- providing assistance to educators which helps transfer the outcomes of research and development to improved practice in local schools and teacher education institutions

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ABSTRACT

This working paper is a product of a graduate research seminar in mathematics education offered in the Department of Curriculum and Instruction during the spring semester 1976-77 at the University of Wisconsin. The seminar was directed by Professor Thomas Romberg. Suppport for organizing the results of the seminar into a working paper was provided by the Wisconsin Research and Development Center for Cognitive Learning as a report from the Studies in Mathematics Project also directed by Professor Romberg.

The graduate seminar decided to focus its inquiry on research toward a particular problem area; namely, the teaching of initial fraction concepts to young children. The reason for this is threefold:

1. A review conference had been held in December 1976 on the teaching of fractions in Developing Mathematical Processes (Romberg, Harvey, Moser, & Montgomery, 1974, 1975, 1976). The conference report (Harvey, 1976) was used as a starting point for the discussions.
2. To those familiar with the teaching of elementary mathematics, common fractions has long been a problem area in part because there are so many different conceptions of how one might approach instruction of common fractions.

3. With the advent of the hand-held calculator and the growing use of the metric measurement system in schools, the purpose and rationale for teaching common fractions and the organization of instruction need to be re-examined.

The rationale for re-examining fractions is discussed in some detail in Chapter I. In Chapter II a logical, conceptual hierarchy of initial fraction concepts is presented. Chapter III is an extensive review of related research. The students in the seminar examined the recent studies, interpretations, and recommendations about the teaching of common fractions. In Chapter IV a brief review of how fraction concepts are presented. Finally in Chapter VI an overview of the DMP approach to fractions is included.

It should be noted that while the various sections of this working paper were initially prepared by the students taking the course a project assistant, Maurice Burke, was responsible for organizing the material and preparing most of the working paper. Time was made available for another student, Nancy Hiber (through the generosity of the Chairman of the Department of Curriculum and Instruction), to work on sections of the working paper during the summer session following the research course.

In all, the paper represents a careful re-examination of initial instruction of fractions. This synthesis of the teaching of common fractions should provide a starting point for a sequence of research studies.

I

INTRODUCTION

Questions dealing with the teaching of fractions form one of the problem spots in today's elementary school mathematics. Many of these questions find their source in two circumstances: (1) the existence of different interpretations of rational numbers and different models to represent these interpretations; and (2) the immanent use of calculators and the metric system in schools.

DIFFERENT INTERPRETATIONS OF FRACTIONS

Many textbooks differ from each other in the interpretation of and the emphasis placed on fractional concepts. Furthermore, most textbooks introduce several interpretations of fractions. In some cases this is very easy to do and can be done almost inadvertently in a single lesson. Botts (1968) provides a good example of how this might be done. He distinguishes three different definitions of fractions:

1. Fractions are numbers expressed in the form _a/b_ where _a_ and _b_ represent whole numbers and _b_ does not equal zero.
2. Fractions are pairs of numbers--a numerator and a denominator.
3. Fractions are symbols or ways of writing numerals.

Botts then reviews twelve recent textbooks (he doesn't disclose which) and finds that four of them present fractions as numbers, at least six present them as number pairs, and at least five present them as symbols. Such usages as "Add 3/2 to 2" treat fractions as numbers; others such as "What is the numerator of the fraction 3/2?" treat fractions as pairs of numbers; such questions as "What number is written above the bar?" treat fractions as symbols with bars in them. All of these usages seem natural in a course of instruction on fractions. In fact, Botts feels that teachers should employ all three with children without any explanation whatsoever, since it is usually completely clear from the context which of the three is being employed (p. 219). The point, however, is that it is very easy to slip from one usage of fractions to another in the course of a single lesson on fractions, and there are textbooks that do so.

Some textbooks, on the other hand, are very deliberate in drawing distinctions between different interpretations of fractions. Van Engen (1960) differentiates fractions from rate pairs in a very precise manner and this distinction is found embedded in the elementary textbook series Seeing Through Arithmetic (Hartung et al.) which he coauthored. In that series, fractions are used to represent part-whole situations while rate pairs are used to express comparisons between two disjoint quantities. These interpretations are developed independently and yet concurrently. Possible confusion arises because fractions and rate pairs both have in common their notations, their membership in only one equivalence class, and their criteria for equivalence.

Kieren (1976) presents perhaps the most exhaustive analysis of the different interpretations given to fractions considered as rational numbers. He outlines seven basic interpretations, their mathematical and cognitive demands, and their function in the broader mathematics curriculum. A brief summary of these interpretations follows.

1. Rational numbers as fractions: Rational numbers can be considered as numbers expressed in the form $\underline{a}/\underline{b}$ where $\underline{a}$ and $\underline{b}$ are integers and $\underline{b}$ does not equal zero. The important mathematical structures implicit in this interpretation are specific algorithms focusing on the standard operations on fractions and on determining equivalence. The corresponding cognitive structures would be a set of skills for using those algorithms. The prerequisites for these skills would be skill in computation with whole numbers and not developed concepts of part-whole relationships and proportionality.

2. Rational numbers as equivalence classes of fractions: Here a rational number is interpreted as an equivalence class of ordered pairs of integers (the equivalence relation being defined as $\underline{a}/\underline{b} = \underline{c}/\underline{d}$ if $\underline{ad} = \underline{bc}$). The principal mathematical ideas underlying this interpretation are those of equivalences and of ordered pairs as elements in a number system. To develop the notion of operations consistent with this interpretation the child must be able to assign a pair of numbers to a part-whole situation in both the discrete case and the continuous case. In this way, the child is led to view the different elements of the equivalence classes as alternate ways of representing the same number. From this, along with the notion of an ordered pair, the child develops the concepts of operations. An ability to handle class inclusion and perform partitionings is assumed.

3. *Rational numbers as ratios*: In this interpretation, rationals are numbers of the form p/q where p and q are integers and q does not equal zero. They are used to represent proportions or comparisons between integers. The basic mathematical structures involved are the notions of equivalence classes and proportionality. The cognitive demands are the ability to handle equivalence symbolically and to transfer number line concepts to ratio numbers. For example, the child must be able to scale a number line in any number of ways. Part-whole or class inclusion notions are not central to this interpretation.

4. *Rational numbers as operators or mappings*: Here, the rational number is thought of as an operator either mapping a finite set onto another finite set or as mapping the Euclidean plane into itself. For example, one can think of the rational number p/q as a transform or operator changing a geometric figure into a figure p/q times as big. Multiplication of fractions finds a natural representation in the composition of two operators, and addition can be interpreted as the sum of two operators [i.e., if $a/b(x)$ represents the results of a/b operating on some number x, then $(a/b + c/d)\ (x) = a/b(x) + c/d(x)$]. On the basis of these definitions, the rationals as a field of operators are constructed. The fundamental mathematical concept underlying this interpretation is that of an operation. The cognitive demands are that the child acquire the notions of proportion, composition (seeing one transformation followed by another as a single transformation), identity, and inverse. Underlying the latter notion would be a generalized

notion of reversability. For example, the child would come to know that each operation of a/b on an element x can be reversed by some operation b/a on a/b(x).

5. Rational numbers as elements in a quotient field: In this interpretation, fractions are used as elements in a quotient field. Put in simplest terms, these elements are quotients of a divided by b where a and b are integers and b does not equal zero; i.e., these elements are the solutions of equations of the form ax = b where a and b are integers and b does not equal zero. Field axioms are assumed and the definitions of addition and multiplication follow from the definition of the elements of the field. The mathematical structures underlying this interpretation involve the notions of an abstract deductive system, problem solving, and quotients. The cognitive demands include deductive reasoning, partitioning, and understanding part-whole relationships. For example, 3(x) = 5 can be represented by the situation of dividing five cakes equally among three children to determine how much (i.e., x) each child will get.

6. Rational numbers as measures: In this interpretation, rational numbers are viewed as points on a number line. Fundamental to this interpretation is the notion that the unit for the number line can be divided into any number of congruent parts. The rational number a/b can be viewed as a measure, i.e., a of b congruent parts. The main cognitive structures required are the notion of a unit and its arbitrary subdivision. The child must realize that a unit is invariant under partitioning and that one can partition a unit into any number of

congruent parts. The child must also recognize part-whole relationships and equivalent situations (one-half of the pie is the same as three-sixths of the pie). Finally, the child must acquire the notion of an order relation.

7. *Rational numbers as decimal fractions*: In this interpretation, a rational number is any number which can be expressed as a terminating or repeating decimal. The cognitive demands are similar to those mentioned for measurement. Here, however, there is more emphasis on the ability to generalize in the symbolic demain; i.e., the child must be able to generalize the decimal numeration system and its algorithms to the fractional situation. Another requirement is that the child be able to attach a standardized description to part-whole situations. Measuring and estimating abilities are also important.

Kieren feels that all of the interpretations have their important uses--some for practical applications, such as rational numbers used as fractions, ratios, decimals, or measures; and some for their applications to later mathematics such as the equivalence class, quotient field, and operator interpretations of rational numbers. Hence, he feels that it is important for children to be given experience with most, if not all, of the interpretations.

In addition to the different interpretations outlined above, there are different ways in which these interpretations can be represented to children. For instance, under the equivalence class and the measurement interpretations of rational numbers, fractions are represented in terms of the part-whole relationship, i.e., *a/b* represents *a* of b equal parts of a whole that has been divided into b equal parts. However, under these interpretations, the part-whole relationship can have several different embodiments: (1) the whole considered as a particular continuous quantity such as a region, a line, or a volume; (2) the whole considered as a set of discrete objects and the parts considered as subsets; and (3) the whole considered as a unit of measurement with respect to either discrete or continuous quantity attributes (numerosity, area, length, etc.) and the parts considered as sub-units of the unit of measurement.

Given the diversity of available interpretations and representations, many questions face the math educator. Which interpretations of fractions should be presented to children and at what stages in their education? Should different interpretations be presented concurrently or at different stages (say in different grades)? Do certain interpretations interfere with others thereby confusing the children? Does the representation (area, set, etc.) used to introduce fractions affect the child's ability to learn the concepts? Do some of these representations induce transfer to related situations better than others? In addition to these questions, there is the consideration of fractional operations to be introduced. Some operations are easier to motivate under some interpretations than under others. Thus, how should the operations be introduced and, indeed, which operations should be introduced? The same considerations and questions apply to algorithms. Which algorithms, if any, should be introduced? Do some interpretations lend themselves to the introduction of certain algorithms better than others? How should algorithms with fractions be explained and motivated?

These are the main questions that have emerged in response to the existence of multiple interpretations and representations of fractions. Some of them are substantive questions which can only be answered by research. Others are policy questions which can be decided only within the context of the educational objectives established for mathematics instruction in general and fractions in particular.

THE USE OF CALCULATORS AND THE METRIC SYSTEM

The answers to policy questions relating to fractions are conditional upon the consideration of the issues raised by the emergence of the use of hand-held calculators and the metric system in the schools. Smith (1974) asserts that most of the need for common fractions has arisen through our use of the English system of measurement. The widespread use of the metric system, though not precluding the need to learn and use common fractions, does place much more emphasis on the use of decimal fractions; since sub-units and multiples of the basic units of the metric system are related by powers of 10. The Nacome Report (1975) concludes with a strong challenge to educators with respect to the teaching of fractions:

> The challenge to traditional instructional priorities is clear and present. If mathematics education takes full advantage of the new technological capabilities, we envision at least the following kinds of changes in school programs.

> First, the elementary school curriculum will be restructured to include much earlier introduction and greater emphasis on decimal fractions, with corresponding delay and deemphasis of common fractions notations and algorithms. This change is appropriate to match the language of instruction to the language of calculators (decimals). But further justification lies in the impending metric system adoption, the opportunity to exploit natural similarities between decimal fractions and whole number algorithms, and the chance to avoid very difficult (and less effectively taught) procedures involving common fractions. (L.C.D., equivalent fractions, and reduction to lowest terms). [Page 41]

Thus, educators are faced with the following questions. Should the decimal interpretation of fractions be presented first and other interpretations later? Should instruction in decimals begin earlier than is now done? To what extent should we teach the other (non-decimal) interpretations of fractions and do we need to teach children the rational number operations using common fraction algorithms? Can decimal fractions be effectively taught before teaching common fractions? What initial fractional concepts are required in order to teach decimal fractions? And finally, how can we teach decimal fractions effectively any earlier than it is now done?

Research on the issues raised by this situation has been scant. Some results have been reported however. Willson (1972) compared a fractions-then-decimals sequence with a decimals-then-fractions sequence of instruction. He found no significant difference in achievement. This supported a similar finding by Faires (1963). This limited evidence cannot be taken as conclusive in any sense on any of the main issues arising from the proposed changes in the elementary education curriculum. More work needs to be done.

The work by Willson, however, leads to certain questions that relate to the issues raised with respect to the interpretations of fractions. Willson used fifth grade students and did not carefully control for possibly influence of prior knowledge of fractions on the treatments he used. By the fifth grade, even without formal instruction, many children have already developed understandings of some basic fraction concepts. Would a decimal-then-fractions sequence compared to a fractions-then decimal sequence have results in no significant differences if they had been introduced in the third grade level? Critical here is the need to know what fraction concepts children have developed prior to formal instruction and what initial fraction concepts must be developed before a child can adequately understand decimal fractions. Such prior knowledge, or lack thereof, could interact differentially with the different treatments.

These considerations generalize to the other interpretations of fractions. Before research can make significant progress on the spectrum of questions concerning fractions there is a need to analyze in more detail the understandings required under the various interpretations of fractions and to determine what fractional concepts children possess at various ages prior to formal instruction on fractions. Work done in these areas would help determine possible areas of commonality in the intital stages of the various interpretations and help assess the various areas of possible interference between the different interpretations. Such work would also be useful in helping to determine optimal sequencing for the different interpretations and to determine which modes of representing fractions (area, set, line, etc.) would be best for initial fraction instruction given to children in light of their prior knowledge of fractional concepts.

The rest of this document will be devoted to work that has been done this past year on the needs mentioned above. The area of concern has been limited to the part-whole interpretation of fractions. This interpretation is implicit in Kieren's "measurement" interpretation and partly involved in the "equivalence class" and the "decimal" interpretations. Within the part-whole interpretation, attention has been focused on initial fraction concepts--those fraction concepts prerequisite to the introduction of the operations on fractions. The objective was to begin a status study on children's understanding of intial fraction concepts under the part-whole interpretation. This required the following:

1. a conceptual analysis of the understandings involved in the part-whole interpretation of fractions;
2. an examination of current textbooks to help determine what interpretations children are being exposed to and what representations are being used to introduce the part-whole interpretation;
3. a survey of research and other writings on the question of initial fraction concepts; and
4. the design and execution of several pilot studies relating to various aspects of initial understandings of fraction held by children.

As a final consideration, the DMP treatment of the initial concepts of fractions was examined and criticized in the light of the Conference on Fractions held at the University of Wisconsin, in December 1976.

The following document has been organized according to the above breakdown of required tasks.

II

HIERARCHY OF INITIAL FRACTION CONCEPTS AND SKILLS

INTRODUCTION

This chapter presents a hierarchy of initial fraction concepts. It is not intended to be a teaching hierarchy. It is a logical, conceptual hierarchy of initial fraction concepts and is bounded by several parameters. One could, however, use it for teaching purposes.

Restrictions on the Hierarchy

The hierarchy of initial fraction concepts was developed within the parameters of our interest as a group. We were concerned with looking at the fraction concepts, but we did not concern ourselves with rational numbers. Further, we limited our investigation to several interpretations of fractions: (A) subset/set, (B) part/whole, and (C) sub-unit/unit (a measurement interpretation of fractions).* Extensions of these three major interpretations included applications of the initial fraction concepts, such as measurement of time, use of rulers, interpretations of dials, and so on. The hierarchy included abstraction to the extent of writing fractions, but it did not attempt to go as far as computation with fractional numerals. We did not consider decimal fraction notation.

---

*In the introduction we spoke of a general "part-whole" interpretation of fractions which included all three of the above. Very often the interpretation (outlined in the introduction) in which the whole is a single object is called the "part-whole" interpretation. This is what is intended in (B) above.

The Origin of the Hierarchy

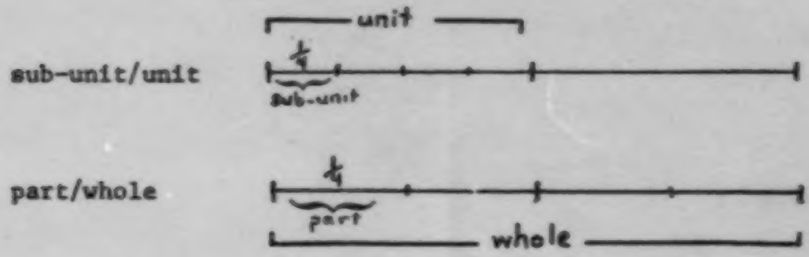
The hierarchy was developed to be as complete as possible for a logical hierarchy, yet it is not as detailed as a Piagetian hierarchy would be. The works of Novillis (1976) and Harvey, Green, and McLeod (1975) as well as the Initial Fraction Sequence developed at Michigan were very useful in the early stages of conceptualization.

DEFINITIONS

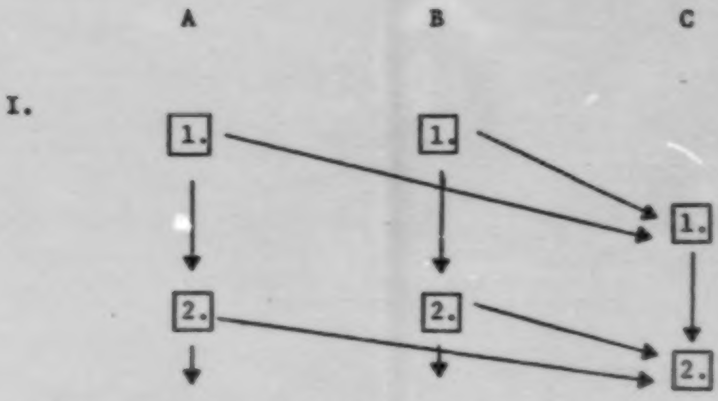
Column A: subset/set. The subset/set interpretation of fractions compares the number of elements in a subset of a set to the number of elements in the whole set. This interpretation could be used exclusively in teaching initial fraction concepts by following the hierarchy straight down Column A.

Column B: part/whole. The part/whole interpretation of fractions compares a number of congruent parts of a whole to the total number of congruent parts that would constitute the whole. This interpretation includes such kinds of wholes as regions, lengths, and volumes. The part/whole interpretation could be used exclusively in teaching initial fraction concepts by following the hierarchy straight down Column B.

Column C: sub-unit/unit. The sub-unit/unit interpretation of fractions compares a number of sub-units to the number of sub-units required to make up the unit. It is distinguished from the interpretation of part/whole because there may be many units that make up a whole thing. The sub-unit/unit interpretation is the measurement interpretation. One need not know what the "whole" is; one need only know the size of the unit, and the sub-unit, in order to measure to fractional dimensions. Example:



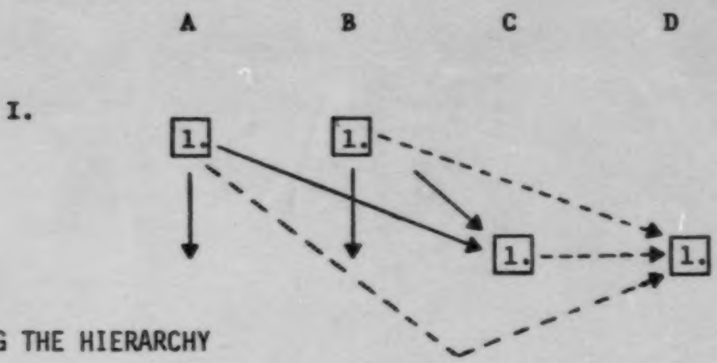
The sub-unit/unit track requires the skills of both part/whole and subset/set in our hierarchy. For this reason, one would teach each preceding concept from Column A and Column B before teaching a concept in Column C. Example:



Column D: extensions and applications. Extensions and applications of initial fraction concepts may vary from another way of showing the subset/set concept, to an application of sub-unit/unit in measurement.

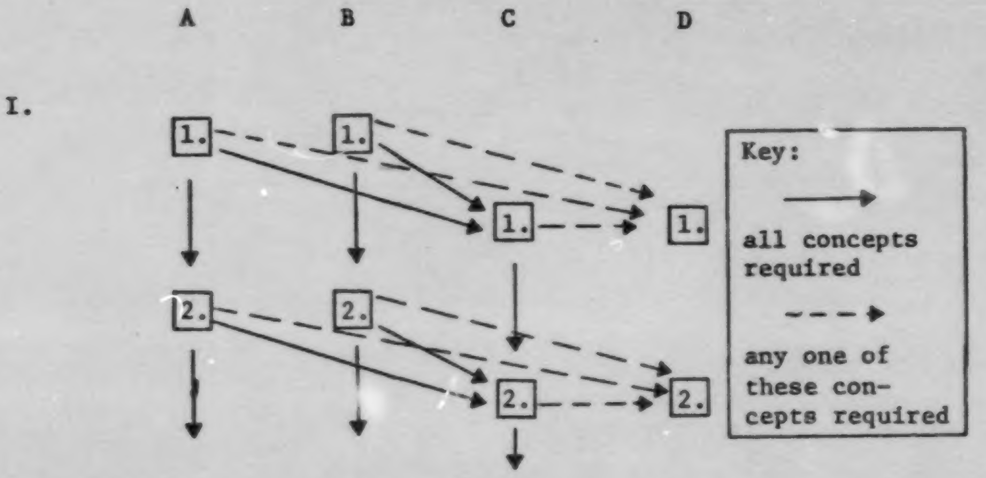
For example: What is the probability of drawing one black ball? Here the set/subset interpretation can be used to represent the probability as a fraction.

Extensions and applications may occur in the hierarchy following a concept in Column A, Column B, or Column C. These "or" prerequisites are indicated with dotted lines in our hierarchy to show that an item in Column D does not require all of the concepts which could lead to it. Example:



FOLLOWING THE HIERARCHY

A schematic representation of the hierarchy would be as shown:


A
B
C
D
I.
1.
1.
1.
1.
2.
2.
2.
2.
Key:
all concepts required
any one of these concepts required

As can be seen, concepts in Column C require all previous concepts from Columns A, B, and C. Column D may include concepts as extensions of A, B, or C, but need not require previous Column D concepts.

CHART 1

HIERARCHY OF FRACTION SKILLS

A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
I. Prerequisites to learning the concepts of fraction.			
1. Counting: associate numeral to set in order to represent the numerosity of the set.	1. Identify attribute (length, area, volume).		
2. The meaning of "same number as" "greater than" "less than" "equal"	2. The meaning of "same length as"--line "same area as"--region "same amount as"--volume		

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
3. =, <, > Use of relevant symbols.	3. The meaning of "greater than" "less than" "equal" with respect to these attributes.	1. Meaning of "unit" "The same unit as"	
		2. Measurements using various units—standard and non-standard.	

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
II a. Distinguish parts from wholes: Break wholes into parts; concrete manipulations.			
1. Divide set into subsets.	1. Divide whole into parts (line, area, volume, concrete quantities (candy bars, etc.), and diagrams.		
		1. Given "unit" divide into "part of a unit."	Time: Given "hour," break into "minutes."
		2. Measurements with such answers as "six sticks and part of a stick."	

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A	B	C	D
Subset/Set	Part/Whole	Sub-unit/Unit	Extensions and Applications
II b. The whole is the union of its parts: Given the parts, reconstruct the whole.			
1. Given "subsets," reconstruct "set."	1. Given "parts," construct "whole" (line, area, volume, concrete quantitatives, diagrams).		
		1. Given "parts," construct "unit."	Time: Given minutes, construct "hours."

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
III. Equivalence partitionings: Parts equal, parts congruent; recognition, then physical construction			
1. Given an example, recognize that a set is divided into subsets with the "same number" in each.	1. Given an example recognize that a line (region, quantity) is divided into parts each having the same length (size).		
2. Given elements, physically divide a set into subsets with the	2. Given a line (region, etc.) physically divide it into parts, each		

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A	B	C	D
Subset/Set	Part/Whole	Sub-unit/Unit	Extensions and Applications
"same number in each."	having the same length (size).		
		1. Given an example of a unit and sub-units, recognize that the unit has been divided into parts of the "same size."	
		2. Given a unit, physically subdivide it into sub-units of	

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
		the same size, so that the sub-units add up to some whole number of these units.	
		3. Measurements using units and sub-units e.g., 2 quarts and 3 pints.	

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

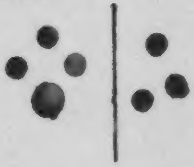
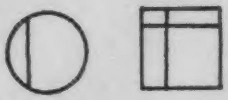
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
IV. Equivalence partitions.			
1. Given examples of non-equivalence partitioning, recognize the non-equivalence, when there are different numbers or objects in each subsets. 	1. Given an example of a region which has been subdivided into non-equivalent parts, recognize the non-equivalence. 		

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

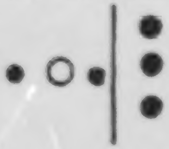
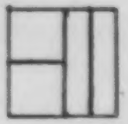
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
2. Given examples of non-congruent equivalence partitioning, where elements of subset are same in number but different in size, recognize that the subsets are equivalent (in number) subsets. 	2. Given a region (line, etc.) which is subdivided into non-congruent but equivalent parts, recognize the equivalence.  Note: Children do not need to be able to reason at		

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
	this stage that halves are equal. Rather, examples should be provided in which children are able to determine that the pieces are the same size by some means besides looking at them.		

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
V. Reconstruct the whole, given an equivalence partitioning.			
1. Given a certain number of subsets, and the knowledge that the whole set has been divided into n parts and each part is equivalent to each other part, determine how many more of these subsets are needed to make the whole set.	1. Given a certain number of congruent parts of a circle, apple, or other recognizable figure, determine how many more of these parts are needed to make the whole.		

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
	2. Given a part (of a circle or recognizable figure), reconstruct the whole.		
		1. Given the sub-units and knowledge that the unit has been subdivided into n sub-units, reconstruct the unit.	

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
VI a. Initial fraction instruction; Presymbolic; Use of words.			
1. Given a set partitioned into n congruent subsets, recognize that it has been divided into nth's.	1. Given a whole (area, length, volume, quantity) divided into n congruent parts, recognize that it has been divided into nth's.		
2. Given a set that has been divided into four unequal subsets or three	2. Given a whole (line, area, volume, quantity) that has been		

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
equal subsets, for example, recognize that it has not been divided into fourths.	divided into four unequal parts or three equal parts, for example, recognize that it has not been divided into fourths.		
		1. Given common units divided into n equal sub-units, recognize that the unit has been divided into nth's.	

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
VI b. Initial fraction instruction; Presymbolic; Use of words. Unit fractions.			
1. Given a subset/set, recognize a congruent subset part compared to the whole, and be able to say "one fourth of" the whole (or designate such a subset as "one fourth of the ___").	1. Given a part/whole example with congruent parts, recognize a part compared to a whole, and be able to say "one fourth of the whole (or designate such a part by writing "one fourth of the ____").	1. Given a sub-unit and the unit, recognize the sub-unit compared to the whole and be	

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A	B	C	D
Subset/Set	Part/Whole	Sub-unit/Unit	Extensions and Applications
		able to say "one fourth of" the unit.	
VII. Identifying instances and use of symbols. Counting fractional parts.			
1. Given a subset/set example with several subsets marked off, be able to count the subsets circled and write "3 fourths" and "2 thirds."	1. Given a part/whole example, where several parts are shaded, be able to count the parts and designate the shaded quantity, as, e.g., "3 fifths" "2 fourths."		1. Given an example, be able to make a part/group comparison using the words "is like." ooo set A oooo set B "Set A is like $\frac{3}{4}$ of set B."

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

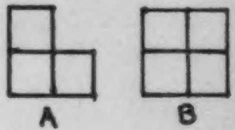
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
		1. Measurements using fractional parts of the unit, e.g., 5 inches and 3 fourths of an inch.	
2. Learn to write, e.g., "3/4" to represent "3 fourths" understanding that the four stands for the number of equivalent subsets the set has been	2. Learn to write, e.g., "3/4" to represent "3 fourths" understanding that the four stands for the number of equivalent parts that the whole has	2. Measurements using mixed numbers to the nearest fourths, thirds, or halves, e.g., 5-3/4 inches.	2. Given an example, be able to make a part/whole comparison, using the words "is like."  "Figure A is like 3/4 of figure B."

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
divided into and the three stands for the number of subsets under consideration (circled, etc.).	been divided into and the three stands for the number of parts under consideration (shaded, etc.).		
VIII. Non-congruent fractional parts.			
1. Examples of non-congruent fractional parts.	1. Examples of non-congruent but equal fractional parts.		1. Examples of non-congruent and non-equivalent fractional parts.

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

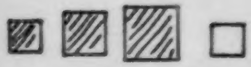
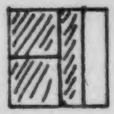
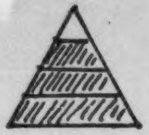
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
			
3/4 of the objects are shaded.	3/4 has been shaded. Note: Child should have some easy way of determining whether parts are equal sized or not.		3/4 is not what is shaded.

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
IX. Constructing fractional parts, given the whole and 1/b.			
1. Given a set, divide it into b equivalent subsets (or bth's). (1/b)	1. Given a line segment, area figure, or volume figure, divide it into b fractional parts (bth's). (1/b)	1. Given a unit, divide it into b sub-units (bth's). (1/b)	
2. Given a set, determine some fractional part of it, e.g.,	2. Given a whole (line, area, volume, quantity), designate some	2. Given a line segment with an initial point, show the point, e.g.,	1. Given a fraction, represent it with a diagram showing the whole and parts

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

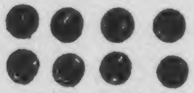
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
"circle 3/4 of the dots." 	fractional part of it, e.g., color 2/3 of the square.	3-3/4 inches from the initial point.	considered. Realize that many different diagrams can be used to represent the same fraction.
X. Interpreting whole (or more than a whole), given a fractional part.			
1. $b(1/b) = 1$, where b is the number of subsets.	1. $b(1/b) = 1$, where b is the number of parts.	Note: In Columns A and B of this section the child is not expected to understand the symbolic representation as shown there. Rather, the child is to learn, for instance, that 3 thirds of a given set or region constitute the whole.	

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
2. Given a/b, where a is the number of subsets shaded, and b is the total number of subsets in the set, realize that (b-a)/b added to a/b makes the whole set.	2. Given a/b, where a is the number of parts shaded, and b is the total number of parts in the whole, realize that (b-a)/b added to a/b makes the whole.	1. b(1/b) = 1, where b is the number of sub-units.	
		2. Given a/b, where a is the number of sub-units shaded, and b is the total number of sub-units in the unit, realize that (b-a)/b added to a/b will make the whole unit.	1. Interpret improper fraction a/b, where a>b, a is the number of shaded parts, and b is the total number of parts into which the whole is divided.

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A	B	C	D
Subset/Set	Part/Whole	Sub-unit/Unit	Extensions and Applications
		3. Understand that mixed numbers, 1-1/2 inches for example, can also be represented in the improper fraction form 3/2, where the two represents the type of sub-unit (half inches) and the three represents how many of the sub-units are under consideration.	

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

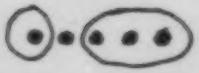
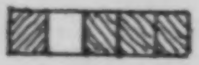
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
XI. Combining and separating fractional parts.			
1. Given an example of a/b and c/b, where $a+c<b$, be able to put together the parts and call it "four fifths." 	1. Given a part/whole example of a/b and c/b, where $a+c<b$, be able to put together the parts and name the result "four fifths." 		

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
"One-fifth put with three-fifths will make four-fifths."	"One-fifth put with three-fifths will make four-fifths."		
2. Given an example of a/b and c/b, where $a>c$, be able to show c/b parts separated from a/b parts and call it "two-fifths."	2. Given a part/whole example of a/b and c/b, where $a>c$, be able to show that c/b parts separted from a/b parts will yield "two-fifths."		

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

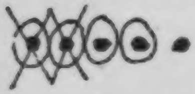
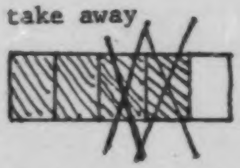
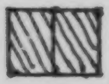
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
 "Four-fifths, take away two-fifths, will leave two-fifths."	take away  leaves two-fifths: 		
		1. Given a sub-unit/unit example of $\underline{a}/\underline{b}$ and $\underline{c}/\underline{b}$, where $\underline{a}+\underline{b}<\underline{b}$, be able to put together the parts and name the results "four-fifths."	1. Given an example from columns A, B, or C, where the sum of the sub-units $\underline{a}$ and $\underline{c}$ is greater than $\underline{b}$, be able to say that this will yield an improper fraction.

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

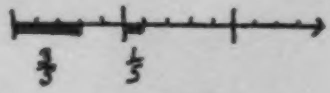
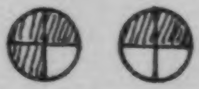
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
		 four-fifths 2. Given a subunit/unit of a/b and c/b where a>c, be able to show that c/b parts separated from a/b parts will yield "one-fifth".	 five-fourths 2. Given an example from columns A, B, or C, be able to separate and name the fraction, in the instance where a>c, and a-c>b.

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

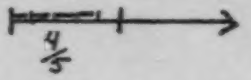
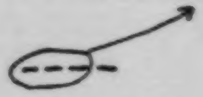
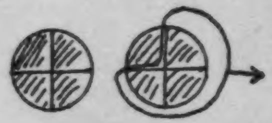
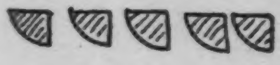
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
		 separate 3/5  will yield one-fifth	 "Eight fourths, separate three-fourths leaves five-fourths." 

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

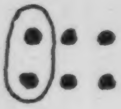
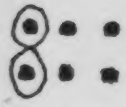
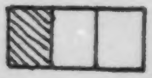
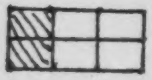
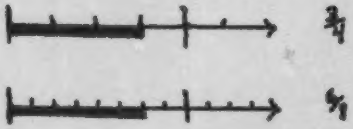
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
XII. Recognition of equivalent fractions.* Visual support.			
1. Given a subset/set example of 1/3 and 2/6, be able to recognize that both represent the same fractional part of the whole.  1/3  2/6	1. Given a part/whole example of 1/3 and 2/6, be able to recognize that 1/3 represents the same region as 2/6.  1/3  2/6	1. Given an example of the sub-unit/unit, be able to recognize that 3/4 represents the same	* The sets, wholes, and lines used should be congruent as though one were the image of the other.

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
		distance as 6/8. 	

XIII. Equivalent fractions (no visual support). Be able to recognize equivalent fractions.

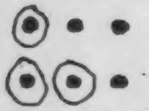
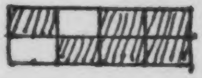
A	B	C	D
1. Given  3/6 and	1. Given  6/8 and		

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

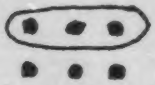
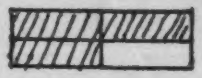
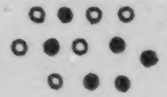
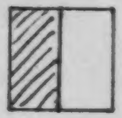
A	B	C	D
Subset/Set	Part/Whole	Sub-unit/Unit	Extensions and Applications
 1/2	 3/4		
recognize that 3/6 is the same as 1/2.	be able to recognize that 6/8 is equivalent to 3/4.		
			1. Given  and 

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

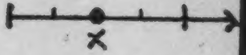
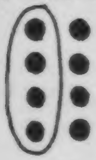
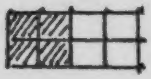
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
			and  be able to recognize that all of these can be used to represent 1/2.
XIV. Construct equivalent fractions: Figures.			
1. Given  1/2 are circled	1. Given:  1/2 of the rectangle is shaded.		

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

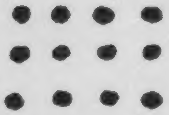
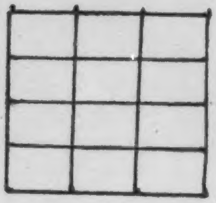
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
be able to determine other fractions that can be used to represent the number of dots that are circled (4/8, 2/4).	be able to determine what other fractions can be used to represent the amount of the whole that is shaded.		
2. Given a figure (e.g., a set of dots) 	2. Given a figure such as 	1. Using a ruler or some other measuring device determine other fractions of a foot, e.g., that are the same length as 1/2 of a foot.	

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
use it to answer such questions as "is one-half of the dots the same as 2/4 of the dots?" "is 3/6 of the dots the same as 3/4 of the dots?"	use it to answer such questions as "is one-half the same as 2/4?"	2. Use a ruler or some other measuring device to answer such questions as "is 1/2 of a foot longer than 4/6 of a foot?"	
XV. Ordering fractions and comparing. Recognition and use of "greater than" and "less than" in fractional setting.			
1. Given that d is the sub-unit and	1. Given a part/whole example of		

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

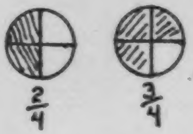
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
$\underline{b}<\underline{c}$, be able to recognize that $\underline{b}/\underline{d}<\underline{c}/\underline{d}$. Use examples. Be able to say 3/6>2/6. ●○○ $\underline{b}=2$ ●○○ $\underline{d}=6$ ●○○ $\underline{c}=3$ ●●○ $\underline{d}=6$ 3/6>2/6	fractions with the same denominator and unequal numerators, e.g., $\underline{b}>\underline{c}$, be able to say that $\underline{b}/\underline{d}>\underline{c}/\underline{d}$. Given these drawings, be able to say that 3/4 is greater than 2/4.  3/4>2/4	1. Given the two drawings showing the same unit and a coordinate at 1/3 on one line, and a coordinate at 2/3	

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

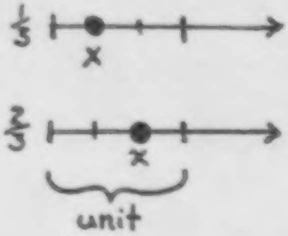
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
		on the other, be able to recognize that 2/3>1/3.	
2. Recognize that fractional parts of sets can be compared and ordered according to numerosity.	2. Recognize that fractional parts of a whole can be compared and ordered according to size.	 2/3>1/3	
3. Given that the unit is the same, the denominators are the same, and the numerators	3. The same as that indicated in Column A except represented by continuous wholes.		

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

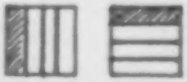
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
are the same in a visual example, be able to recognize that the fractions represent the same number. $\frac{1}{3}$ ● ○ ○ $\frac{1}{3}$ ○ ● ○			
1/3 = 1/3	1/4 = 1/4	2. The same as that indicated in Column A except represented by number line.	

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

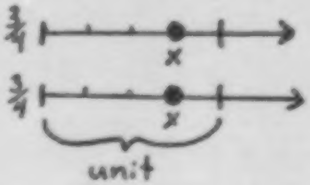
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
		 3/4 = 3/4	
4. Given that the unit is the same, the denominators are different, and the numerators are the same in an example, be able to recognize	4. The same as that indicated in Column A except represented by continuous wholes.		

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

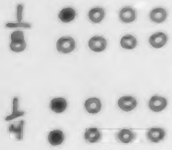
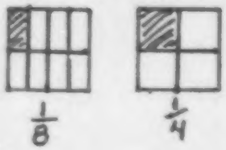
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
that the fraction represented with the smaller denominator is the larger fraction. 			
1/8 < 1/4	1/8 < 1/4	3. The same as Column A except represented by the number line.	

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

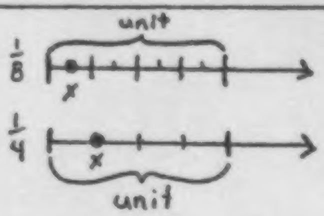
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
		 1/8 < 1/4	
5. Given that the unit is the same, the denominators are different, and the numerators are different in an example, be able to order the fractions.	5. The same as Column A except represented by continuous wholes.		

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

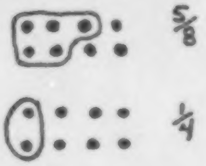
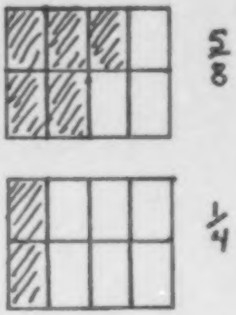
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
(Here we assume the fractions are such that one denominator is a multiple of the other.)  1/4 < 5/8	 1/4 < 5/8	4. The same as indicated in Column A except represented by number line.	

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

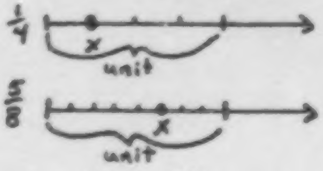
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
		 $1/4 < 5/8$	
6. Realize that the numerosity of a fractional part depends on the unit it is a fractional part of.	6. Given that the units are different, the numerators are the same and the denominators are the same, be able to recognize that the region is larger in the example with the		

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

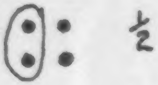
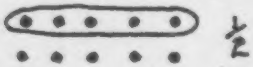
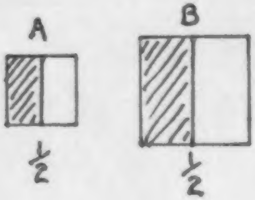
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
A.  B.  1/2 of Set A contains less dots than 1/2 of Set B.	larger unit, although the fraction numerals are the same.  The region in example B is larger.		

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

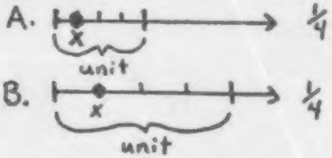
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
		5. The child recognizes that the relative size of the subunits depends on the size of the units.  1/4 of a unit in A is smaller than 1/4 of a unit in B.	

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

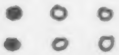
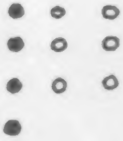
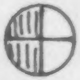
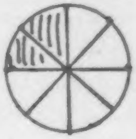
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
7. Given the examples in which the unit is different, the numerators are the same, and the denominators are different, recognize that the fractions are not easily compared.  $\frac{2}{6}$  $\frac{2}{5}$	7. Same as Column A except represented by continuous wholes.  $\frac{2}{4}$  $\frac{2}{8}$		

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

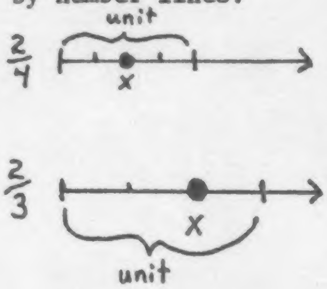
A	B	C	D
Subset/Set	Part/Whole	Sub-unit/Unit	Extensions and Applications
Understand that in comparisons involving fractions, it is crucial to keep in mind the whole that is implicitly refered to by the fraction.		6. Same as Column A except represented by number lines.  	

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

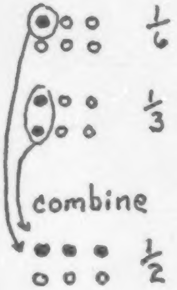
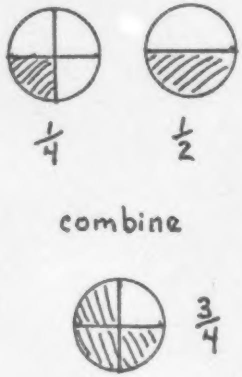
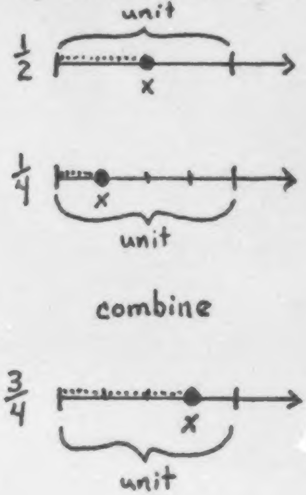
A Subset/Set	B Part/Whole	C Sub-unit/Unit	D Extensions and Applications
XVI. Combining fractional parts (unlike denominators).			
1. Given fractional parts using the same unit, be able to recognize that it is possible to combine fractional parts to get another fraction.  $\frac{1}{6}$ $\frac{1}{3}$ combine $\frac{1}{2}$	1. Same as in Column A except represented by continuous wholes.  $\frac{1}{4}$ $\frac{1}{2}$ combine $\frac{3}{4}$		

CHART 1, CONTINUED

HIERARCHY OF FRACTION SKILLS

A	B	C	D
Subset/Set	Part/Whole	Sub-unit/Unit	Extensions and Applications
		1. Same as Column A except represented by number lines.  unit $\frac{1}{2}$ x $\frac{1}{4}$ x unit combine $\frac{3}{4}$ x unit	

III

ABSTRACTS OF RELATED RESEARCH

This chapter contains abstracts of some of the important works on fractions. The research included herein has been limited to that concerning initial fraction concepts, readiness for learning fractions, sequencing of initial fraction concepts, and issues of representations. Research on operations and algorithms, though abundant, has not been included. Several graduate students contributed abstracts and will be identified appropriately. Abstracts without author credits were prepared by the present authors. A brief summary of some of the more pertinent results will be included at the end of the chapter. The abstracts have been ordered according to the following categories:

1. Overviews of Research
2. Dissertations
3. Large scale Investigations (e.g., NLSMA study)
4. Foundational work--Piaget et al.
5. Further studies on concepts held by children and readiness
6. Interpretations, representations, and recommendations

OVERVIEWS OF RESEARCH

1. Using Research: A Key to Elementary School Mathematics—Rational Numbers: Fractions and Decimals. By Marilyn Suydam and J. Fred Weaver.

2. An Interpretive Study of Research and Development in Elementary School Mathematics. By Marilyn Suydam and Alan Riedesel.

3. Suggestions from Research. By Rosemary C. Anderson.

4. Research on Rational Number Learning. By Thomas Kieren.

USING RESEARCH: A KEY TO ELEMENTARY SCHOOL MATHEMATICS--RATIONAL NUMBERS: FRACTIONS AND DECIMALS. Suydam, M., & Weaver, J. F. Columbus, Ohio: ERIC Information Analysis Center for Science, Mathematics and Environmental Education, 1975.

The research reported in this summary is presented according to how it relates to questions that are posed by the reviewers. The following questions and suggestions concerning initial fraction concepts are put forward:

1. Can young children learn fractional concepts? The reviewers find from surveys of what children know about mathematics upon entering school, that at least 50 percent can recognize halves, fourths, thirds, and have acquired some facility in using these fractions. They cite research that indicates that children in second grade can be taught intial fraction concepts effectively. They conclude, however, that a planned systematic program for developing fractional ideas seems essential as readiness for work with symbols. Use of manipulative materials appears vital in this preparation.

2. What sequence should be used in teaching fractions? The reviewers cite Novillis' result that associating fractions with part-whole and part-group models was prerequisite to associating a fraction with a point on a number line. They then cite results from the Michigan studies which recommend that a content sequence begin with work on equivalent fractions using diagrams and sets of objects and proceed to addition, and then multiplication of fractions. They recommend that in developing fractional operations, conceptual work for comparing fractions using diagrams should precede presentation of the rule or algorithm.

The other questions to which the reviewers address the research are as follows:

1. What procedures are effective in work with addition and subtraction with fractions?
2. How can we most effectively develop the algorithm for multiplication with fractions?
3. What algorithms shall we use for division with fractions?
4. Is it helpful to analyze the errors that pupils make with fractions?
5. Is it helpful to relate decimals with fractions or place value?
6. How should we teach children to place the decimal point in division with decimals?

Note: A useful, rather up to date, bibliography is included.

AN INTERPRETIVE STUDY OF RESEARCH AND DEVELOPMENT IN ELEMENTARY SCHOOL MATHEMATICS. Suydam, M., & Riedesel, A. Childhood Education, January, 1971.

The authors summarize their review with a series of questions, and asks: What concepts can be taught the preschool child?

Several of their comments relate specifically to fractions and measurement. First of all, a large percentage of the kindergarten children tested were familiar with measurement terms and instruments. Secondly, on fractions, about half of the kindergarten children tested were able to recognize half of an item. In other tests, 89 percent were able to recognize thirds of an item and 67 percent, fourths. In another study 78 percent recognized halves, 51 percent recognized thirds, and 50 percent recognized fourths.

SUGGESTIONS FROM RESEARCH. Anderson, R. C. Arithmetic Teacher, 16(2), 131-135.

This article includes a good bibliography of journal articles on fractions published prior to 1969. It contains a summary of research findings in the form of suggestions for teaching. Some of these suggestions are relevant to initial fraction concepts:

1. By the first grade children understand unit fractions through halves and fourths as applied to single objects.
2. To improve teaching, increased emphasis should be placed on developing basic concepts, teaching the relationships which exist among topics, and making use of concrete experiences with real objects.
3. The following recommendation is included in the section on the fractional operations: The intuitive basis for addition of fractions which do not have the same denominator can be established meaningfully through the use of rectangular strips of paper.

RESEARCH ON RATIONAL NUMBER LEARNING. Kieren, T. Paper prepared for Study Group C at the Third International Congress On Mathmatics Education, Karlsrurhe, Germany, August, 1976.

Kieren starts by giving a brief statement of the needs and issues in rational number learning.

1. Adolescents and adults have not learned rational number concepts very well.
2. There are strong mathematical demands on children trying to learn these concepts, e.g., sophisticated application of the proportionality schema and a general notion of inverses.
3. Rationals occur in a wide manner of interpretations.
4. Formal instruction has done relatively little to enrich the experience base that children need to develop rational number concepts.

On the basis of this, the author states the long term goal for the Rational Number Group (at the University of Georgia) as the establishment of a theory of concept development and tested curriculum experiences for children ages six to fifteen, with regard to rational number concepts. He sets out the three questions that guide the research of the Rational Number Group:

1. What is meant by the conceptual framework of the rational numbers in childhood and adolescence?
2. By what cognitive mechanisms does such a framework develop?
3. What are instructional procedures which promote this development?

He includes a section on paradigms for research and gives a general schema by which different research studies on rationals can be categorized and related.

Under the review of research the author sees several important results and developments:

1. Michigan studies have uncovered conflicts that arise when instruction based on different interpretations (e.g., ratio and measurement) are used together.
2. Sambo found that it was typical of textbooks for nine, ten, and eleven year olds to present fraction concepts in a great variety of modes simultaneously, probably leading to confusion on the part of the children.
3. Owen's result is suggestive of a hypothesis that the attainment of the structures of the full stage of concrete operations may be a necessary condition for learning initial rational number concepts.
4. Sambo and Babcock are further exploring this issue of the relationship between measure and rational number acquisition.
5. Other research (Novillis) shows that up to the age of twelve to thirteen children have great difficulty disassociating the notion of a unit from "everything there."
6. Kieren's own work on the operator interpretation of fractions suggests that this notion develops through several stages and only at eleven to twelve years of age are children able to work with non-unit operators. Even fourteen year olds may not have acquired a concrete concept of composition.

DISSERTATIONS

1. Learning the Basic Concepts in Fractions and Their Applications in the Addition and Subtraction of Simple Fractions. By Sister Mary Theodine Sebold.

2. A summary of some of the Michigan studies:
   a. A comparative study for the learning structures of three algorithms for the division of fractional numbers. By J. K. Bidwell.
   b. A comparison of two approaches, area and finding a part of, and two instructional materials, diagrams and manipulative aids, on multiplication of fractional numbers in grade five. By G. A. Green.
   c. The effects of placement of rules and concrete models in learning addition and subtraction of fractions in grade four. By L. Ellerbruch.
   d. The effect of algorithmic and conceptual development for the comparison of fractions. By S. A. Choate.
   e. The effect of the ratio approach and a regional approach on equivalent fractions and addition/subtraction for pupils in grade four. By T. G. Coburn.

3. An Investigation of the Learning of the Initial Concepts and Oral/Written Symbols for Fractional Numbers in Grades Three and Four. By C. Muangnapoe.

4. Achievement of Fractional Number Concepts, Ages Six Through Ten, and on Decimals, Eight Through Ten. By P. J. Galloway.

5. A Comparison of Two Conceptual Frameworks for Teaching the Basic Concepts of Rational Numbers. By D. Sension.

6. The Child's Understanding of Three Interpretations of Certain Unit Fractions Prior to Formal Instruction. By B. G. Campbell.

7. An Analysis of the Fraction Concept into a Hierarchy of Selected Subconcepts and the Testing of the Hierarchical Dependencies. By C. F. Novillis.

LEARNING THE BASIC CONCEPTS IN FRACTIONS AND THEIR APPLICATIONS IN THE ADDITION AND SUBTRACTION OF SIMPLE FRACTIONS. Sebold, M. T. *The Catholic University of America Educational Research Monographs*, October 15, 1946, *14*(2).

Abstract prepared by Maurice Burke

Purpose: The problem of how children learn fractions had apparently not been investigated. The literature in this field seemed to indicate that children do not "learn" fractions. To teach fractions successfully teachers must know the mental processes that children use while attempting to learn them. This study was undertaken for the purpose of discovering what these processes are, i.e., those that lead to an understanding of the basic concepts of fractions. The main questions posed were:

1. What have children in grades three through eight learned about fractions?
2. What processes do they go through in learning fractions and what difficulties do they encounter?

Procedure: Data were collected from three sources:

1. *Individual testing and interviewing*, where 100 children from grades three through seven, from four schools in Washington, D.C., were individually tested and interviewed (20 minutes to an hour).
2. *A group test*, in which 600 children (100 from each of the grades from three to eight from three schools in Wisconsin) took a 44 item test constructed by the author to test fractions.

3. Individual instruction, where 24 fifth graders were taught individually the basic concepts of fractions up to and including addition and subtraction of fractions. The instruction consisted of ten lessons, an hour each.

Results and Conclusions: For grades three and four: The group test data showed that only a very small percentage of the pupils in the third grade could name the fractional part of a figure that was equally divided and that about one-fourth of them recognized that unequal divisions of figures could not be designated as fractional parts. For some, any part of a figure was one-half, to others it was a quarter, while still others alternated their answers between one-half and a quarter, irrespective of the number of parts into which the figure had been divided. Another group named the fractional part according to the number of parts into which a figure was divided, irrespective of equal or unequal divisions.

About one-third of the pupils in the third grade could find the fractional parts of groups of figures with reference both to unit and proper fractions. But an understanding of the concept was possessed by far less than a third of these pupils. In general, unit fractions were handled better than other proper fractions.

For about 50 percent of the pupils in the third and fourth grades, the word fraction connotated "one-half."

In comparing fractions, about 60 percent of the pupils in the third grade knew that one-half is greater than one-fourth. But to them one-sixth was larger than one-third and also larger than one-half because "six is larger than three" and "larger than '2'." The performance of the pupils in the third grade on the comparison of other unit fractions and proper fractions was almost nil.

The interview data show that in reality the only fractions that most of the pupils in the third grade could add were one-half and one-half.

Besides the scanty knowledge of the fact that a fraction may be one or more of the equal parts of a single object, some pupils in the third grade also recognized two other uses of a fraction; namely, its being one or more of the equal parts of a group and of a number.

The data of the concepts of fractions known by the pupils of the fourth grade differed from those presented by pupils in the third grade in that twice as many children in the fourth grade could name the fractional parts of an equally divided object and also recognized that unequal divisions of an object were not fractional parts having the same denominator. The pupils of the fourth grade had a more extensive knowledge than those of the third grade in comparing fractions, a small percentage of the former being able to show clearly that one-third and one-half were larger than one-sixth, three-fourths larger than one-half, and two-thirds larger than one-third. In addition and subtraction of fractions the fourth grade pupils surpassed those in the third grade in that the former could add a few more fractions than the latter and could likewise subtract some fractions--which the third grade pupils could not do.

For the grades five through eight:

Concept 1: *A unit fraction is one of the equal parts of a whole.* Over half of the students in these grades knew how to apply unit fraction designations to diagrams of part-whole relationships. However, interview data showed that very few of the pupils in the fifth grade and only one-fourth of those in the sixth and seventh grades could give evidence of a clear understanding of the concept.

Concept 2: *A fraction may also be more than one of the equal parts of a whole*. Over three-fourths of the students could pick out the fraction that represented some partly shaded diagram of a part-whole situations but less than half of them could partition square into designated fractional parts. From individual instruction it was evident, however, that this concept was not difficult for the majority of the children after they had learned the concept of a unit fraction.

Concept 3: *A fraction may be one or more of the equal parts of a group*. The results show that this concept was understood by approximately two-thirds of the pupils in the fifth grade; by three-fourths of the pupils in the sixth and seventh grades; and by 85 percent of the pupils in the eighth grade. But the data obtained by interview were not so encouraging. Less than a third of the pupils in the fifth grade and about half of those in the sixth and seventh grades furnished evidence of a clear understanding of the concept.

Concept 4: *A fraction may be one or more equal parts of a number*. The percentages of correct responses indicated a fair mastery of this concept by pupils in grades six, seven, and eight.

Concept 5: The concept of numerator and denominator. The results of the test items seem to indicate that the learning of the concepts numerator and denominator had been mechanical rather than for understanding.

Concept 7: The concept of an improper fraction. Over three-fourths of the students could change improper fractions into mixed fractions, and change mixed fractions back into improper fractions. But less than a fourth could associate an improper fraction with a picture, perhaps because they failed to see the unit. A similar result holds for the concept of a mixed number though there was more success (about 50 percent) on using a mixed number to describe a diagram.

Concept 10: The concept that a fraction may express comparison between single objects, groups, or numbers. Less than half of the students in grades five through seven could determine the proportion between two numbers. In eighth grade almost three-fourths could. The teaching experiment showed the learning of this concept was difficult.

The results of the individual testing and of the individual instruction programs are as follows:

1. There is much evidence of rote learning in fractions among both bright and dull children.
2. The vocabulary of fractions is difficult. Most of the children do not have an adequate background to acquire an adequate understanding of the fractional terms.
3. The fundamental processes of addition, subtraction,

multiplication, and division of integers are not yet "meaningful habituations" for all pupils of the fifth grade and hence cause interference in the learning of fractions.

4. In general, the learning of the basic concepts in fractions progresses through the following levels of understanding:
   a. No knowledge of the concept.
   b. Erroneous ideas of the concept.
   c. Confusion of the concept with others, especially with those which have been only partially learned.
   d. Partially correct but vague ideas.
   e. Knowledge that an incorrect presentation or illustration of the concept is not true.
   f. Recourse to the visualization of a previously acquired model with descriptive phrases explaining it.
   g. Memorized information of the concept, with or without understanding.
   h. Partial understanding.
   i. Full understanding.

Abstractor's Notes: The study is an older study in which there seems to be a large concern for the use of language, e.g., "numerator," "denominator," and "equals," as though a child's not knowing these words constituted some basic lack of understanding of the concepts. The study attempts to be comprehensive in its examination of the learning of fractions. Because of the large number of concepts being tested for, and the relatively small number of items, and because of the ambiguous nature of many of the items, one can raise some questions about how accurately the group test reports the children's understanding of the concepts. The author acknowledges this on the basis of the discrepancy between the interview results and the group test results.

A SUMMARY OF SOME OF THE MICHIGAN STUDIES.

Abstract prepared by Charles Mitchell

Bidwell, J. K. A comparative study of the learning structures of three algorithms for the division of fractional numbers, DAI, 1968, 29, 830A.

Green, G. A. A comparison of two approaches, area and finding a part of, and two instructional materials, diagrams and manipulative aids, on multiplication of fractional numbers in grade five, DAI, 1970, 31, 676A-677A.

Ellerbruch, L. The effects of placement of rules and concrete models in learning addition and subtraction of fractions in grade four, DAI, 1976, 36, 6441A-6442A.

Although these studies are chiefly concerned with approaches or algorithms dealing with fractions and one of the four basic operations, they do provide data on the effect of initial fraction representations on the development of algorithms. They lend more serious consideration to the questions: Does one choose the most appropriate approach to an algorithm based on the conceptual work that has been completed, or vice versa?

Choate, S. A. The effect of algorithmic and conceptual development for the comparison of fractions, DAI, 1975, 36, 1410A.

This study compared four approaches to teaching comparison of fractions, varying the time of presentation of a comparison algorithm.

One sequence was the Rule Algorithm in which fractions were compared using a rule without any conceptual development. This approach was superior to the others in providing a better base for solving proportions in number sentence form.

The Meaningful Algorithm sequence attempted to connect conceptual development and algorithmic ideas in each step. Requiring subjects to learn two things at the same time and integrate them in a short time appeared too demanding as this treatment produced the weakest result.

The third and fourth sequences emphasized conceptual work with either a late introduction of an algorithm or no algorithm at all. These treatments were superior in enabling subjects to better transfer learning to the untaught operations of addition and subtraction.

Coburn, T. G. The effect of the ratio approach and a region approach on equivalent fractions and addition/subtraction for pupils in grade four, DAI, 1974, 34, 4688A-4689A.

The purpose was to investigate two author-designed instructional sequences for equivalent fractions in grade four. The Ratio Sequence was based on an initial ratio thinking model, while the Region Sequence was based on a region thinking model which emphasized paper folding activities.

Although the Ratio Model was shown to be effective as a means of developing the renaming generalizations with grade four subjects, Coburn recommended not using it. Based on the results and experience with the author-taught classes, it seemed apparent that the Region Model built a foundation which provided for a smoother connection with adding and subtracting unlike fractions, continuing maintenance for renaming in higher terms, and a smoother transition to a part-part ratio interpretation for the fraction symbol.

AN INVESTIGATION OF THE LEARNING OF THE INITIAL CONCEPTS AND ORAL/
WRITTEN SYMBOLS FOR FRACTIONAL NUMBERS IN GRADES THREE AND FOUR.
Muangnapoe, C. Doctoral Dissertation, University of Michigan, 1975.

Abstract prepared by Maurice Burke

Purpose: This study was designed to obtain information about the teaching and learning of the initial concepts and symbolisms for fractional numbers using the Initial Fraction Sequence which was developed under the direction of Dr. Joseph Payne at the University of Michigan. The major questions posed were:

1. How effective is the IFS on pupil achievement, retention, and transfer in classroom and small group settings?
2. How does achievement of pupils in IFS compare with results from other populations of selected items from some standardized tests?
3. What are the main difficulties pupils encounter in learning the initial fraction concepts?
4. What revision should be made in the IFS as a result of this study?

Rationale: It is generally conceded that there is a need to provide children with more experience with the fundamental interpretations of fractions and their representations in concrete situations before going on to their more formal aspects. However, there are many ways of interpreting fractions and no small amount of variety in the way they are used. On examining the different interpretations and representations of fractions, it seems reasonable to say that the very basic concept of fraction is that it is a part of a continuous whole. Thus there is a need to develop and test curricula dealing with this initial stage of the fraction development using the part-whole interpretation with emphasis on continuous wholes.

Design and Procedure: The Initial Fraction Sequence was taught to two third grade classes and two fourth grade classes (106 pupils in all) taken from a school in Ann Arbor, Michigan. The classes were taught by teachers from the school while 15 of the children were taught by the investigator in groups of three. Eight lessons were taught, one per day, each lasting between 30 and 45 minutes. Instruction was aimed at pupils achieving mastery of each lesson of the sequence.

Achievement measures included an author-designed posttest, a retention test, and a transfer test. The posttest included items taken from the National Longitudinal Study of Mathematical Abilities (NLSMA). The transfer test included items dealing with applications, fractional parts of a set, fractional parts of regions, and addition of fractions. A pretest consisting of approximately one-half of the items of the posttest was also given.

Analysis of variance and analysis of covariance using the fraction pretest as the covariate were used to compare results between the small groups and the classes and between pupils in the IFS and pupils in the NLSMA study. Item and category analyses were used to identify the major difficulties encountered by students.

Findings and Conclusions: The pretest and posttest showed no significant differences between the small groups and the classes. However, significant differences did occur favoring the small groups on the transfer and retention tests. Significant gains were made in pupil performances from the pretest to the posttest on the 30 items in common.

On the posttest 64 percent and 73 percent of the pupils attained, respectively, a 90 percent and 80 percent level of achievement. The pupils in the IFS scored significantly better than seventh grade students of the NLSMA study on selected items taken from the NLSMA tests.

The children had the most difficulty with the following areas:

1. Identifying a unit when many units were present. This was especially evident in the recognition and identification of a unit on a number line.
2. Realizing the need for equal-sized parts of a unit.
3. Comparison of fractions using fractional symbols without the help of diagrams.
4. Dealing with fractions greater than one--recognizing, identifying, and comparing them.
5. Applying fractions to a number line.

Results further suggested that the Region Model was easier for children than the number line model. Most problems on fractions under the Region Model came from items using fractions greater than 1 in mixed or improper form. Children also had greater difficulty in writing a fraction that would describe a given diagram than in determining whether a given fraction described a given diagram.

In line with the above difficulties, the following revisions were recommended for the Initial Fraction Sequence:

1. More emphasis should be placed on identifying the unit among many units and associating whole numbers with them. A greater variety of units should also be used.
2. More emphasis is needed on partitioning units into equal-sized parts and more examples of non-fractional partitionings should be provided.
3. More background work with number lines using whole numbers should be provided before the number line is used to represent fractions.
4. Fractions greater than one should be introduced early, in the form of mixed numbers, with many experiences being provided in concrete measurement situations. Improper fractions should be introduced later as alternate ways to represent mixed numbers.
5. More emphasis should be placed on size comparisons using diagrams, and the connection between these and fractional symbols should be stressed.

In conclusion the Initial Fraction Sequence is an effective instrument for instruction and can be used with success by classroom teachers. It compares well with other fraction sequences.

Abstractor's Note: The study is useful in pointing out the difficulties that children sometimes encounter in learning fractions. It provides a carefully planned and tested set of lessons on fraction concepts that can be mastered by third and fourth grade students. There is some question about the results in that the items testing for the perception of the unit in a fractional situation were not very clear. This ambiguity in the presentation of the problem very likely caused some confusion and thus led to the poor performance of the children on recognizing the unit in a fractional situation. This raises a doubt about one of the author's conclusions that the failure to recognize a unit was the main cause of difficulty in learning fractions. It would be interesting to see how second grade students would perform if given the Initial Fraction Sequence. The Initial Fraction Sequence is summarized below.

Initial Fraction Sequence

Lesson 1: The children measure areas and lengths using sheets of paper (all the same size) and sticks (all the same size) as units. Measurements are made to the nearest whole number of the units. However, children are encouraged to distinguish cases where "part" of a unit is needed, either saying: "Three and part of a unit" or "small part of a unit less than four."

Lesson 2: The children cut objects, sheets of paper and sticks into equal-sized pieces. They get practice folding paper into two, three, four, five, six, eight, and ten equal parts. (Circular pieces of paper are also used in the partitioning activities.) The children then compare parts of a unit divided into, say, three parts with parts of that unit divided into two parts.

Lesson 3: The children learn to use fractional names for equal-sized parts of units. They use oral and written names such as "seven-eigths."

The child learns that a sheet of paper cut into four equal parts has been cut into fourths and that each part is called a fourth. The child also learns that he or she can count fractional parts and have two-fourths, three-fourths, four-fourths and even five-fourths (using a neighbor's sheet of paper). Fractions from halves through tenths are used and the children learn how many of the various fractional parts make up a whole.

The children then compare halves and fourths of different units in order to realize that one-half of some unit may be larger than one-half of another unit. They then compare different unit fractions of the same unit. Finally they use their paper and sticks to compare fractions with the same denominators, e.g., two-sixths is less than four-sixths.

Lesson 4: The children learn to write fraction symbols for the oral and written word names and for concrete representations of fractions. They learn that in three-fourths the four means that "there are four equal parts in a unit" and the three indicates that we take three parts into consideration. Given a fraction the children represent it with a piece of paper. Given a concrete representation the children identify the fraction that represents it. This procedure is repeated for fractions equal to and greater than one.

Lesson 5: Here the children learn to use diagrams to represent fractions. They learn that the diagram is to be considered as a picture of some unit. They partition diagrams (regions and line segments) into equivalent parts and give those parts fractional names.

Lesson 6: The children divide diagrams of various shapes into fractional parts. They then make comparisons between the unit and proper fractions using the diagrams.

Lesson 7: The children are introduced to using mixed forms such as 2-1/2 to represent the measure of an area or a line segment. The children draw diagrams to represent mixed forms and answer questions such as how many wholes and halves cover a region. They then use paper rulers to measure line segments into wholes and halves.

Lesson 8: The children learn to associate fractions in mixed form with points on a number line. A segment is divided into a whole number of units and then fractional parts of the unit are introduced onto the line.

Abstractor's Note: The present author agrees with Muangnapoe that mixed forms should be introduced before improper fractions with the latter being introduced as alternate names for the former. The part of the sequence dealing with fractions greater than one has some ambiguous examples and questions which might be remedied by pointing out to the children what the unit is in the particular situation. Time spent on sevenths and ninths might be utilized to reinforce other understandings (work on number lines for instance).

ACHIEVEMENT OF FRACTIONAL NUMBER CONCEPTS, AGES SIX THROUGH TEN, AND ON DECIMALS, EIGHT THROUGH TEN. Galloway, P. J. Doctoral Dissertation, University of Michigan, 1975.

Abstract prepared by Maurice Burke

Purpose: The purpose of this study was to investigate the learning and teaching of initial fraction concepts and notations at various age levels from six to ten and the learning and teaching of decimal concepts and notation at ages eight, nine, and ten. The main questions posed in the study were the following:

1. What parts of the Initial Fraction Sequence can be learned by children of ages six through ten?
2. What parts proved easy or difficult to learn?
3. How does achievement and time needed for mastery on fractions compare at various age levels?
4. Is the percentage of retention of fraction concepts the same for the various age levels?
5. At which age level can decimals be introduced and mastered?

Design and Procedures: The Initial Fraction Sequence (IFS) was taught during a two week period at Newport School in Ann Arbor, Michigan. The study involved 250 students, ages six through ten, and nine classroon teachers. (The sequence was the same as that used by Muangnapoe.) The five primary classes were regrouped into five groups according to criteria of age and mathematical achievement. The four upper primary classes were kept intact. A three-day decimal sequence was taught following the fraction sequence. All fourth and fifth grade students participated in the decimal unit and remained in their regular math groups for the instruction. Twenty-six bright third graders were taught by the investigator. The decimal sequence involved the following three objectives:

1. To name fractions and mixed forms for tenths using decimals.
2. To write a fraction or mixed form and decimal for length on a ruler marked into tenths.
3. To read a centimeter ruler to the nearest centimeter and to the nearest tenth of a centimeter and to measure segments to the nearest centimeter.

A pretest was given to all the grades. Different posttests were given to primary grades one, two and three than were given to the upper primary grades, four and five. Retention tests and attitude tests followed later. The posttests had 19 questions in common covering the main objectives of the sequence. On the basis of these, comparisons were made between the upper primary and the primary grades.

Results and Conclusions: With respect to question (1) the students of all age levels were able to correctly identify a unit and equal-sized parts. Eighty percent of all age levels could also identify fractions less than one. All age levels could use concrete models and diagrams effectively in identifying fractions. Students in the upper elementary grades were able to master most of the sequence, but had some difficulties working with number lines. The same holds for the upper ability groups of the primary level. The middle ability groups of the primary level were able to master the first four units of sequence and make significant progress toward mastery (in the two week period) of units five, six, and eight. The lower ability group (mostly first graders) were able to master the first three units. They had difficulty using (writing and reading) fractional symbolisms and thus did not receive instruction on the entire IFS.

All age levels found difficulty with the number line representation of fractions. Test items which involved any writing or diagramming by students proved to be difficult for primary students. The trend seemed to demonstrate this latter difficulty as developmental difficulty rather than a problem with some concept to be learned.

When looking at all five age levels, there is a gradual growth in achievement with significant leaps occurring between first and second grade and between third and fourth grade.

The results of the decimal posttest indicate that students in the third grade and above can master the basic decimal concepts and symbols. In both grades three and four, 81 percent of the involved students attained mastery. Grade five was not significantly higher. The grade three children did show some difficulty in solving the test items dealing with reading and writing decimals from a centimeter ruler.

Recommendations: Primary students are enthusiastic about fractions and are able to master basic concepts. Decimals can also be taught effectively at the primary level (after an initial fraction sequence of some sort). Students should have a longer period of working with concrete materials. Further, drawing and reading of line segments should be introduced in the primary grades.

Abstractor's Note: The study had some limitations that were pointed out by the investigator. First of all, there was a high absence rate during the two week period that the sequence was taught due to an influenza outbreak in the school. Thus, some of the children missed instruction on some of the units. Secondly, the school's population consisted mainly of upper middle class children and hence probably above average students. The results have to be looked on with some caution for these reasons.

A COMPARISON OF TWO CONCEPTUAL FRAMEWORKS FOR TEACHING THE BASIC CONCEPTS OF RATIONAL NUMBERS. Sension, D. Doctoral Dissertation, University of Minnesota, 1971.

Abstract prepared by Maurice Burke

Purpose: The investigation was designed to study the relative effectiveness of three instructional treatments of common fraction concepts. Treatment A used a "region" representation of fractions, Treatment B used a "set-subset" representation of fractions, and Treatment C used a combination of region and set interpretations. Second grade students with no prior formal instruction in fractions were used in the study. The following questions were asked:

1. Was the instruction effective?
2. Are there significant differences between the achievement of the pupils in the three different treatments on items which are consistent with the assigned treatment?
3. Do Treatment A children do significantly better on region items than Treatment B children?
4. Do Treatment B children do significantly better on set/subset items than Treatment A children?
5. Does one treatment produce more generalizability than either of the other treatments?

Rationale: Recent literature has emphasized that mathematical concepts should be embodied in physical situations when they are being taught to children. Some argue that for some concepts children should be provided with "multiple embodiments," in particular, that fractions should be introduced using several interpretations instead of one. The claim is made that this not only will avoid confusing the abstract concept with the physical embodiment but will also improve transfer of abstract concepts to new situations. Still other educators and researchers argue that a multiple embodiment approach leads to confusion and that a single embodiment would be better. Once the student has mastered one embodiment, transfer to others should be easily mastered. A review of some commonly used textbooks reveals that several distinctly different models are being used to introduce the concept of fractions to children. The three most common are the region, set, and number line (length) interpretations. There is a need to uncover evidence concerning the advisability of using each of these "conceptual frameworks" in an instructional setting. Also, there is a need for evidence on the "multiple embodiment" dilemma—i.e., is it better to use one or more interpretations to introduce the concept of fractions to children. Research has been silent, for the most part, on these issues. Most studies reported deal with teaching the computational skills of rational numbers and not with the introductory interpretations of fractions.

Design and Procedure: Children from the second grade classrooms of two Bloomington, Minnesota grade schools were randomly assigned to three treatment groups. The instructional materials were prepared by the investigator. To reduce teacher effects, scripts were provided for the teachers to read to the children during instruction. The children had no prior instruction in fractions. The region and set and combination treatments were nearly identical in terms of examples, number and kind of questions. and even wording. There were eleven lessons lasting about 20 minutes each given on eleven consecutive school days. Student achievement was used to make the comparisons. A domain referenced achievement testing system was used to measure achievement.

The Test: Three item characteristics were crossed to generate 160 item types. They were (1) the numerical value of the item (unit fraction, non-unit fraction, improper fraction, and mixed number); (2) physical model used (Area Model, Set Model, and Number Line Model); (3) task format (seventeen different types of questions, e.g., "given a fraction, select an appropriate pictorial representation" or "given a fraction, produce an appropriate pictorial representation"). An item sampling technique was used so that each child was tested on only thirty items (representative of thirty of the 160 different item types).

Results: Effectiveness of treatments: Using a comparable (control) group of second grade students who had not experienced any of the three treatments to provide baseline data, results show that the experimental groups in all cases produced higher percentages of correct responses than did the control group. On five of the seventeen item formats, the experimental groups performed above the 80 percent level while on these same formats the control group scored below 20 percent.

Comparison among treatments on consistent items: The comparisons between the performance of the region group on the seventeen item formats with area content and the performance of the set group on these same formats with set content, showed no significant differences. In similar comparisons involving the set group with the combination group on item formats with set contents only one comparison was significant (favoring the set group). Likewise, on area items, on only one item format was there a significant difference between the performances of the region group and the combination group.

Comparison between Treatments A and B on area items: On items involving area, comparisons of ten item formats produced no significant differences between the region group (Treatment A) and the set group (Treatment B).

Comparison between Treatments A and B on set items: On items involving sets, of seven comparisons of items formats made, no significant differences were found between these two groups.

Comparison among treatments on number line items: No significant difference was found between the performances of the area group and the set group on number line items. No significant differences were found between the set group and the combination group on number line items. However, a significant trend was found favoring the combination treatment--i.e., nine of thirteen comparisons favored the combination treatment. No significant differences were found on all but two of the comparisons between the region group and the combination group on number line items. In one comparison, the area group was significantly better than the combination group, while in another comparison the combination group was significantly better than the area group. Again there was a significant trend favoring the combination group (on eleven of thirteen item formats).

Conclusions:

1. The experimental materials did produce changes in the behavior of the students. Apparently, these concepts can be taught to second grade students.
2. There are no significant differences among the three treatments in terms of their effectiveness with regard to items consistent with the assigned treatments.
3. Data do not support the hypothesis that there were significant differences between the area treatment and the set treatment as measured by performance on area items, or as measured by performance on set items.
4. There are significant differences between the three treatments in terms of their generalizability to a model other than the ones utilized in the instruction (Number Line Model). This significant difference favors the combination treatment. The data provide support for the "multiple embodiment" philosophy.

Abstractor's Note: There are some limitations that should be borne in mind in interpreting this study:

1. All students involved in this study were enrolled in two

suburban Minneapolis school. Hence it cannot be assumed that the results would generalize to other types of communities.

2. Students used to collect baseline data were not randomly sampled from the same population as those in the experimental treatments. Hence, while the schools and students may have been similar, it cannot be assumed that the baseline data accurately indicated the prior knowledge of the experimental population.
3. Statistical analysis required assumptions (normality, continuity, and equal numerosity of sample) that ruled out many possible comparisons with the data. More appropriate statistical analysis might detect further differences.
4. Transfer results hold only for the Number Line Model.
5. From the comparisons, it is not clear how fraction type (unit, proper, improper, mixed) may have favored the performance of the Area or Set Models groups differentially.

THE CHILD'S UNDERSTANDING OF THREE INTERPRETATIONS OF CERTAIN UNIT FRACTIONS PRIOR TO FORMAL INSTRUCTION. Campbell, B. G. Doctoral Dissertation, University of Oklahoma, 1974.

Abstract prepared by Maurice Burke

Purpose: This study seeks to answer four questions:

1. Prior to formal instruction, what understanding does a five, six, or seven year old child have of the interpretations of fractions as parts of wholes, parts of sets, and an operation of division whereby a whole consists of two one-halfs, three one-thirds, and four one-fourths?
2. Will a child's understanding of a fraction of a set be affected by changing the sizes of the objects of the set?
3. Which unit fraction, one-half, one-third, or one-fourth, does the young child know the most about before formal instruction?
4. Does the young child understand the interpretation of one-half, one-fourth, and one-third better when physical or pictorial representations are used?

Background and Rationale: There have been many recommendations in professional journals and conference reports that experiences with the concepts of fractions ought to be provided for children in K-3. However, researchers are also recommending that teachers build on previous understandings of number concepts wherever they exist. Textbooks include material on fractions very early but it is unclear whether the authors have attempted to base this in any way on what the children already know or intuit about fractions. Children's initial number concepts should be explored so that instructional programs can be adapted to their level of understanding. Research has shown that many young children (ages five to seven) already possess some knowledge of halves, thirds, and fourths. But it is not clear under what interpretations and using what kind of objects the children are most familiar with fractional concepts.

Research Design and Procedure: The population was defined as the students of Lincoln Parish School System, Ruston, Louisiana. The six schools in this system housing kindergarten, first, and second year students were used. Thirty-four kindergarten, thirty-three first year, and thirty-three second year students were randomly selected from these schools. The investigator designed and piloted the tasks. Each task was presented in individual interviews. Tape recordings of the interviews were made to aid in subsequent evaluations. Levels of understanding were rated as follows: (1) clear evidence of understanding, (2) some evidence of understanding, (3) uncertain evidence of understanding, (4) clear evidence of not understanding, and (5) evidence lacking.

Tasks: Tasks were designed to test for understanding of three interpretations of fractions: part of a whole, part of a set, and operation of division. "Apple tasks" were used to deal with concrete representations of the interpretations of fractions while "glass tasks" were used to deal with pictorial representations. The apple tasks used plastic apples, some divided into halves, some into thirds, and some into fourths. For example, children were asked where they would cut the apple so that the investigator would get one-fourth of it (part-whole). Children were asked to pick up one fourth of the apples on the table when there were four apples on the table (set-part). Children were asked to pick up the number of quarter (one-fourths) apples it would take to make a whole apple (division interpretation). Similar format was followed for one-half and one-third tasks as well as the glass tasks in which pictures of glasses, full or partially full, were used.

Results: Interpretation of one-half: Ninety-seven percent of all five, six, and seven year olds showed "clear understanding" of what is meant by one-half of a whole when concrete objects are used. Eighty percent can determine that two one-halves make a whole. Sixty percent seemed to have a clear understanding of one-half of a set when two objects are inside. (It makes little difference whether the objects are the same size or not.) A majority of seven year olds understood the interpretation of one-half of a set when it consisted of four concrete objects. When pictorial representations are used "clear understanding is cut in half." An increase in levels of understanding is objserved with increased age.

Interpretation of one-fourth: Over 50 percent of five, six, and seven year olds are successful in determining that four one-fourths make a whole. If pictorial representations are used 78 percent of the children showed no understanding. One-fourth of a whole was understood very little by any of the age groups represented. Six and seven year olds generally understood the meaning of one-fourth of a set of four elements. (Non-congruence doesn't matter.) Five year olds showed little understanding of this.

Interpretation of one-third: Over 50 percent of the five, six, and seven year olds understood the interpretation that three one-thirds make a whole when concrete objects were used. When pictorial representations were used to test understanding of this interpretation, most children at all age levels demonstrated they have "no understanding." Very little clear understanding is exhibited when questioned about one-third of a whole (concrete or pictorial). More than 54 percent of the six and seven year olds have a "clear understanding" of the interpretation of one-third of a set of elements the same size. Approximately 50 percent of the five year olds showed no understanding of this interpretation.

Conclusions (Prior to Formal Instruction):

1. Half of a whole is understood by more five, six, and seven year olds than either of the other two interpretations.
2. Five, six, and seven year olds understand how many one-thirds, one-fourths, and one-halves it takes to make a whole.
3. Level of understanding of each interpretation increases with age.
4. Children show more understanding of the interpretations of fractions when concrete representations are used than when

semi-concrete representations are used.

5. Children understand one-half better than one-third, and one-third better than one-fourth.
6. The child's understanding of one-third of a set and one-fourth of a set is greater if the objects are different sizes than when the objects are the same size.

Abstractor's Note: The design of the tasks may be responsible for some of the results that seem surprising. There is a big difference between the number of persons who supposedly know that four-fourths make a whole and the number of persons who are able to cut an apple into fourths. In the division tasks the child only had to pick up the particular pieces and see how many it took to make a whole regardless of whether he or she knew that they were thirds, fourths, or whatever. Hence the division tasks are of questionable validity. Also, since the set tasks consisted most of taking a third of a set of three, or a fourth of a set of four, without knowing what the third or fourth meant, a child might get the right answer simply by responding to the cue one-_____ which came before each fractional name. The fact that the children did better on sets with non-congruent objects lends support to this hypothesis. Numerical symbolism was not used for fractions.

AN ANALYSIS OF THE FRACTION CONCEPT INTO A HIERARCHY OF SELECTED SUBCONCEPTS AND THE TESTING OF THE HIERARCHICAL DEPENDENCIES. Novillis, C. F. Journal for Research in Mathematics Education, 1976, 7, 131-144.

Abstract prepared by William Kraus

Purpose: The purpose of the study was to determine the validity of a hierarchy (constructed by the researcher) of subconcepts of the fraction concept.

Rationale: Many studies (and many teachers) have indicated that elementary school students have difficulty developing a fraction concept and, subsequently, learning to compute with fractions. This study looks at the development of a fraction concept in terms of a Gagné-type learning hierarchy "based on (a) mathematical development, (b) sequence in elementary mathematics textbooks, and (c) the limited research that has been done on fraction concepts."

Procedure: The sample studied consisted of 279 students from grades four to six of a predominantly middle-class school. Each student was given a "fraction concepts" test consisting of 16 subtests--one subtest to measure each of the 16 subconcepts identified in the hierarchy being used. Dependency relationships were to be examined between each pair consisting of a superordinate subconcept and its immediate subordinate subconcept. The research hypothesis examined was

> if a student has obtained criterion on the subtest associated with a superordinate subconcept, then he has also

attained criterion on the subtest associated with each immediate subordinate subconcept.

Findings: Out of the 23 dependency relationships examined, 18 were "supported" by the data. Among the hypotheses supported were:

1. associating fractions with Part-Whole and Part-Group Models was prerequisite to associating a fraction with a point on a number line.
2. associating a fraction with a Part-Whole and a Part-Group Model were prerequisite to using a fraction in a comparison situation involving the respective models.
3. associating a fraction with a Part-Whole Model or with a Part-Group Model was prerequisite to associating a fraction with the respective model where the number of parts was a multiple of the denominator and the parts were arranged in an array that suggested the denominator.
4. associating a fraction with a Part-Whole Model or a Part-Group Model having congruent parts was prerequisite to associating a fraction with the respective model having non-congruent (equal) parts.

Interpretations: In addition to several specific conclusions about the hierarchy studied, the researcher concluded that elementary school students are not exposed to a sufficient variety of fraction concepts and, additionally, that not enough negative instances of the fraction concept (e.g., one out of four unequal parts is not one-fourth of the whole) are demonstrated to students.

Abstractor's Note: Novillis's study is of interest primarily because of the hierarchy of fraction subconcepts that she provides. Other researchers may find it quite useful in studying the development of initial fraction concepts. Fairly clear descriptions of the subconcepts in the hierarchy are provided in the article.

Although apparently not based on a random sample, the data gathering and analyzing techniques are interesting. In addition, the data are presented clearly and in some detail and are therefore useful to the reader who wants to "dig a little deeper."

There is one overriding flaw in Novillis's analysis of the data that is quite disturbing as one tries to evaluate the claimed results. Instead of using the data to reject hypotheses, the researcher attempted to use the data to "support" hypotheses. This basic misunderstanding of the hypothesis testing procedures leads to numerous interpretation problems in the study. For example, the researcher uses a sequence of data tests to determine if a hypothesis is supported. In testing the hypothesis "Part-Group Model concepts are prerequisites for Number Line Model concepts" it was found that 78 students met the criterion for neither concept, no students met the criterion for the latter but not the former, 176 students met the criterion for the former but not the latter, and 25 students met the criterion for both. Since the ratios

$$\text{Ratio 1} = \frac{176}{176 + 0}, \qquad \text{Ratio 2} = \frac{25 + 176}{25 + 176 + 0}$$

were greater than .75 and .90, the hypothesis was "supported." This criterion will be met, however, whenever the number of students meeting the subtest criterion for the superordinate subconcept is small relative to the number of students meeting the subtest criterion for the immediate subordinate subconcept regardless of any dependency relationship between the two subconcepts. (Using a similar hypothesis-test, we could conclude that a dependency relationship existed between adding two plus two and writing a Pulitzer prize-winning novel.) And, of course, no amount of further manipulations of the data will ever prove the hypotheses of this study, instead we can only hope to disprove them.

In short, the study is useful not for its specific conclusions, but rather for the potentially testable hierarchy of fraction subconcepts that it presents.

LARGE SCALE INVESTIGATIONS

1. The Grade placement of Arithmetic Topics: A Committee of Seven Investigation. By Carleton W. Washburne.

2. The Work of the Committee of Seven on Grade Placement in Arithmetic. By Carleton W. Washburne.

3. NLSMA Reports: No. 1, Part A; X-Population Test Batteries; and No. 4; Description and Statistical Properties of X-Population Scales. Edited by James W. Wilson, Leonard S. Cahen, and Edward G. Begle.

4. National Assessment of Education Progress, Mathematics Objectives.

5. Mathematics Assessment Results from the Year 04. National Assessment of Educational Progress. By T. Carpenter, T. Coburn, R. Reys, and J. Wilson.

6. Results and Implications of the NAEP Mathematics Assessment: Secondary School. By T. Carpenter, T. Coburn, R. Reys, and J. Wilson.

7. Notes from National Assessment: Addition and Multiplication of Fractions. By T. Carpenter, T. Coburn, R. Reys, and J. Wilson.

TEH GRADE PLACEMENT OF ARITHMETIC TOPICS: A COMMITTEE OF SEVEN INVESTIGATION. Washburne, C. W. Report of the Society's Committee on Arithmetic (29th Yearbook of the N.S.S.E.). Bloomington, Illinois: Public School Publishing Company, 1930, 641-670.

Abstract prepared by Maurice Burke

Purpose: The capacity of a child to learn a given process in arithmetic is a function of his or her mental age and degree of mastery of more elementary facts or processes. Thus the Committee addresses itself to the following questions: (1) Given a mental age as measured by I.Q., when can children learn the airthmetic operations, the meaning of fractions, and graphing? (2) What degree of mastery of previous facts and skills is needed?

Procedure: Using children from 100 different cities the Committee's strategy was basically to determine an approximate grade level for certain arithmetic topics. Those topics were then taught at the determined grade level, at one year lower, and at one grade level higher. The children were given an I.Q. test and a foundations test to check their previously learned skills. On the topic, the children were given a pretest, tests used during instruction, a final exam, and a retention test (six weeks later). Scores from the retention tests were used to draw conclusions about readiness and mastery. Detailed directions on teaching procedures were included.

Results:

1. There is a mental level below which attempts to teach a process are futile.

2. There is a degree of mastery of prerequisites without which the process cannot be efficiently learned.
3. If I.Q. level and foundations results are adequate, then the child has a reasonably good chance of learning new material well and of retaining it.

Results on meaning of fractions: The test consisted of pictures of discrete and continuous fraction situations. For example, in the discrete case the picture might be a set of pennies grouped into piles while in the continuous case it may be a milk bottle or a line. Children were asked questions such as the following: Here are three eggs. Here are nine eggs. Are three eggs 1/2 or 1/4 or 1/3 as many as nine eggs?

The children did much better on the problems involving continuous wholes. In the problems involving discrete grouping situations the children had more difficulty. By the time the mental level of nine years has been reached, a child's chances are three to one in favor of making 80 percent or better on the retention test in the non-grouping problems. For grouping problems, it is necessary to wait until a mental level of eleven years, seven months has been reached before a child has a three to one chance of making a retention test score of 80 percent.

Summary and Implications: The meaning of fractions not involving grouping can safely be taught to children with mental ages of nine years or over with reasonable certainty of success, from the standpoint both of learning and retention. Meaning of fractions involving grouping should, however, be postponed until a mental age of at least ten years, nine months and preferably, eleven years, seven months has been reached.

THE WORK OF THE COMMITTEE OF SEVEN ON GRADE PLACEMENT IN ARITHMETIC.
Washburne, C. W. Child Development and the Curriculum (38th Yearbook
of the N.S.S.E., Part I). Bloomington, Illinois: Public School
Publishing Company, 1939, 299-325.

Abstract prepared by Maurice Burke

In this article, the author summarizes the results of the investigations of the Committee of Seven and responds to several criticisms that had been leveled against the findings. He also points out some of the limitations of their study.

In response to a criticism by Brownell, the author asserts that the Committee's data are in no sense dependent upon any theory of maturing. They merely show that the probability of success in mastering certain topics in arithmetic is greatly increased by waiting until the child has achieved certain prerequisite knowledge and has reached a given mental age. He also agrees that the tests, for the most part, only indicate the level of a child's development at which he or she could complete the learning of a topic with all its difficulties and that, therefore, the time for beginning the instruction was in no way indicated.

Recommendations: Mental age nine to ten: The meaning of simple fractions of a whole object definitely belongs at this level. Children can learn to recognize and distinguish half of an object, a third, a fourth, three-fourths, and so forth. They can also compare length, height, width, and so forth involving the practical concepts of one-half, one-third, and one-fourth.

Mental age ten to eleven: The addition and subtraction of fractions and mixed numbers with like denominators, if confined to those commonly used in life, can successfully be learned at this level.

Mental age twelve to thirteen: The meaning of fractions of a group of objects belongs here, such as the recognition that three objects is a third of nine objects. Following this, children can learn the multiplication and division of fractions. They can draw and measure lines accurately to the eighth of an inch and can calculate the number of inches in a fraction of a yard.

Mental age fourteen to fifteen: The usual textbook problems in addition and subtraction of unlike fractions involving finding a common denominator and, in the case of subtraction of mixed numbers, involving borrowing, are not successfully learned below a mental age of fourteen.

Limitations:

1. A different mode of instruction than that used throughout the studies might have produced different results.
2. Tests all too often didn't test understanding and concepts but rather ability to perform certain operations.

NLSMA REPORTS: NO. 1, PART A, X-POPULATION TEST BATTERIES; NO. 4, DESCRIPTION AND STATISTICAL PROPERTIES OF X-POPULATION SCALES. Wilson, J. W., Cahen, L. S., & Begle, E. G. (Ed.). The Board of Trustees of the Leland Stanford Junior University, 1968.

Abstract prepared by Maurice Burke

This is a summary of results on items dealing with fractions given in the spring and fall for the fourth grade.

Fall Year 1, Grade Four: The following items were administered: (1) 1/4 = 2/4 = _____; (2) 1 - 1/3 = _____. (NLSMA battery, Form 102, Part I, Computation, items 14 and 15.) The following statistics were obtained: Mean number correct (S-Population, 1785 subjects) 0.27

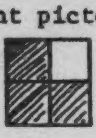
Percentage correct on item one: 17%

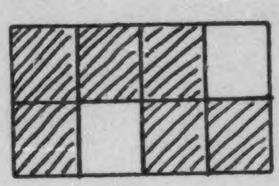
Percentage correct on item two: 10%

Spring Year 1, Grade Four: A six item scale on the subtraction of fractions was included in this battery. (NLSMA Battery, Form 102B, Part I, items 3, 4, 6, 24, 26, 27.) Three of the items had like denominators, two items had denominators where one was a multiple of the other, and one item was a fraction subtracted from 1. Example: 8/9 - 3/9 = _____. With a sample size of 1479 students the following statistics were obtained: Mean number correct out of six items: 2.11

Percentage of correct responses on items with like denominators: 52 - 54%

Percentage of correct responses on items without like denominators: 17 - 18%

Spring Year 1, Grade Four: This Battery also included items on fractional representations (13 items) (NLSMA Battery, Form 102B, Part IV, items 42-54.) The scale was designed to measure recognition of equivalent pictorial representation of the fractional number represented as  and those given in the problems.

Example:  (A) YES (B) NO (C) ?

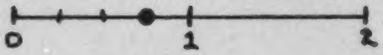
With a sample size of 1479 students the following statistics were obtained: Mean number correct out of thirteen items: 5.99

On items where the whole was divided into four equal parts and three of the parts were shaded 77 - 88% of the subjects responded correctly.

On items in which the whole was divided into four unequal pieces and three of the parts were shaded 23 - 56% of the subjects responded correctly.

On items in which the whole was divided into eight equal parts and six were shaded 12 - 13% of the subjects responded correctly.

On items divided into thirds or sixths 57 - 65% responded correctly.

On number line representations percentages were generally low with only 26% understanding that 

represented three-fourths.

NATIONAL ASSESSMENT OF EDUCATIONAL PROGRESS, MATHEMATICS OBJECTIVES.
Ann Arbor, Michigan: National Assessment of Educational Progress,
1970.

Abstract prepared by Mary Ann Konsin

Ten areas were included in the first assessment: reading, writing (written expression), science, mathematics, social studies, citizenship, vocational education (career and occupational development), literature, art, and music. The purpose of the assessment was to provide helpful information about the progress of education that could be understood and accepted by laymen as well as professional educators. The objectives were established to define a set of goals which were agreed upon as desirable directions in terms of observable behavior. They were acceptable to:

a. scholars in the discipline
b. most educators (as desirable teaching goals)
c. thoughtful lay citizens.

The age groups defined for the assessment were nine, thirteen, seventeen, and twenty-six to thirty-five year olds. Students in parochial, private, and public schools were included. A range of difficulty levels were considered:

a. questions all in the age group should answer
b. questions one-half of the age group should answer
c. questions for the most knowledgeable and highly skilled in the age group.

Three dimensions were defined for the mathematics areas:

a. uses for mathematics

b. mathematics curriculum content areas

c. abilities or behaviors conducive to effective utilization and understanding of mathematics.

A three-fold classification was developed within the dimensions:

1. Social mathematics (needed for personal living and effective citizenship in our society)
    a. reading and use of symbols
    b. basic arithmetic
    c. consumer arithmetic
    d. simple measurement and conversion of units of measurement
    e. ratio
    f. estimation
    g. data interpretation
    h. reading of graphs and charts
    i. intuitive geometry
    j. general logical thinking
2. Technical mathematics (necessary for various skilled jobs and professions beyond that which is needed for personal living and effective citizenship) needed by
    a. carpenter
    b. engineer
    c. physicist
    d. statistician

        e. computer specialist (age seventeen to adult)

    3. Academic mathematics (formally structured mathematics)

        a. sets

        b. field properties

        c. number systems

        d. nature of proof, etc.

Content to be covered in the test, with specific reference to fractions where indicated:

A. Number and Numeration Concepts

    1. Numeration systems

    2. Number systems

        a. rational numbers--representation (fraction, decimal, percent)

B. Properties of Number and Operations

    1. Rational numbers

        a. field properties

        b. order properties

        c. density properties

C. Arithmetic Computation

    1. Rational numbers (positive and negative)

D. Sets

E. Estimation and measurement

F. Exponents and logarithms

G. Algebraic expressions

H. Equations and inequalities

I. Functions

J. Probability and statistics

K. Geometry

L. Trigonometry

M. Mathematical proof

N. Logic

O. Miscellaneous topics

P. Business and consumer mathematics

Q. Attitude and interest items

Objectives determined with respect to fractions and particular age groups:

A. Recall and/or Recognition of Definitions, Facts, and Symbols.

Age 9: Reading and writing fractions

Knowledge of vocabulary

a. numerator

b. denominator

c. proper fraction

d. mixed numeral

Age 13: Knowledge of the rational and real number system

Knowledge of terminology

a. rational number

Not specified for the 17 year olds and young adults.

B. Perform Mathematical Minupulations.

Age 9: Addition and subtraction of fractions having the same denominator.

Changing fractions to higher and lower terms (equivalent fractions)

Multiplication of fractional numbers

Not specified for 13 or 17 year olds, or young adults.

C. Understand Mathematical Concepts and Processes.

Age 9: Showing with diagrams the meaning of a fraction and higher or lower terms

Not specified for the 13 or 17 year olds, or young adults.

D. Solving Mathematical Problems--Social, Technical, and Academic.

Age 9: Using proportions, as in recipes and mixing punch.

Not specified for 13 or 17 year olds, or young adults.

E. Using mathematics and mathematical reasoning to analyze problem situations, define problems, formulate hypotheses, make decisions, and verify results.

F. Appreciation and Use of Mathematics.

Comments: The above reflect the concern of the Committee on Assessing the Progress of Education (CAPE) with respect to the area of mathematics as well as the specific topic of fractions. The three-fold classification seems to imply specific direction for education in terms of mathematics: personal living and effective citizenship, technical achievement, and "academic mathematics." Carrying this to the extreme would lead to a three-tract program, at least within the secondary school curriculum. I feel the first two are contained within the third, implying meaningful learning of mathematics at all levels. Is it possible to satisfactorily test all of these objectives given the time restraints?

MATHEMATICS ASSESSMENT RESULTS FROM THE YEAR 04. NATIONAL ASSESSMENT OF EDUCATIONAL PROGRESS. Carpenter, T., Coburn, T., Reys, R., & Wilson, J. Draft of a report from the NCTM Project for Interpretive Reports on National Assessment, 1974.

Abstract prepared by Mary Ann Konsin

The time limit for the test was:

a. Ages nine, thirteen, seventeen (in-school) 45 minutes

b. Age seventeen (out-school) and young adults (twenty-five-thirty-six) three hours.

There were between 2100 and 2600 respondents per grade level.

The nine year olds were tested in January, February 1973:

a. 1 percent were in Grade 2

b. 25 percent were in Grade 3

c. 73 percent were in Grade 4

d. 1 percent were in Grade 5

The thirteen year olds were tested in October, November 1972:

a. 3 percent were in Grade 6

b. 27 percent were in Grade 7

c. 69 percent were in Grade 8

d. 1 percent were in Grade 9

The seventeen year olds in-school were tested in March, April 1973; out-of-school were tested from May through July, 1973:

a. 2 percent were in Grade 9

b. 12 percent were in Grade 10

c. 63 percent were in Grade 11

d. 11 percent were in Grade 12

e. 12 percent were out of school

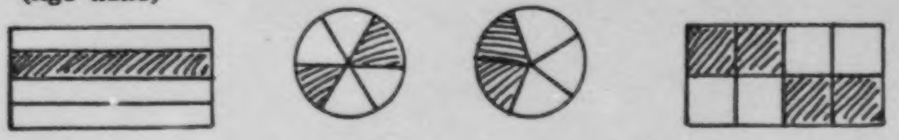
The young adults were tested from October, 1972 through May, 1973. The general conclusions regarding the outcomes on the NAEP questions with respect to fractions are as follows:

a. For the nine year olds: 20 to 35 percent show knowledge of elementary fraction concepts. The others show an absence of fraction concepts rather than erroneous knowledge.

b. The thirteen year olds and young adults performed at about the same level; the seventeen year olds performed slightly better.

The following seven items dealt specifically with the concept of fractions. The items will be listed as well as the age group tested and the results. The label refers to the label used in NAEP.

1. RA 10: What fractional part of the figure is shaded? (Age nine)



Results: Ten to 16 percent of the nine year olds either gave no response or indicated "I don't know (IDK)." Twenty-seven to 46 percent attempted to name the shaded part. The results of this item indicated to the authors two possibilities:

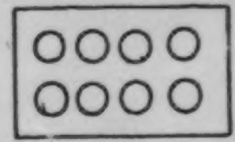
1. Unfamiliar with content
    a. fraction concepts as part of a whole
    b. convention of presenting fractions by diagrams

2. Unfamiliar with language

2. RA 12: A candy bar is broken into three pieces of the same size. Each piece is what part of the candy bar? (Age nine)

   Results: Sixty percent of the respondents indicated unfamiliarity with the fraction concept in this context by replying either with no response or IDK.

3. RC 14: If one-fourth of the dots in the figure are removed, how many dots will be left? (Age nine)



   Results: Forty-nine percent answered four, an indication of their lack of understanding of fraction concepts. Three percent replied IDK.

4. RA 13: There are 13 boys and 15 girls in a group. What fractional part of the group is boys? (Age thirteen, seventeen, adults)

   Results: The results were disappointing: the thirteen year olds and adults performed at about the same level; the seventeen year olds performed slightly better.

   a. 15 to 22 percent no response: IDK
   b. 15 to 17 percent gave the ratio of boys to girls (13/15)
   c. 29 to 44 percent gave another unacceptable response
   d. 20 to 36 percent gave the correct response

   The authors felt the errors were due to a lack of mastery of fraction concepts and application to problem contexts.

5. RB 08: Select the number closest to 3/16. (Age thirteen, 17, adults)

   5/32

   1/4

   5/16

   3/8

   IDK

   NR

   Results: The thirteen year olds performed the poorest on the item, the seventeen year olds the best. The distractor five-sixteenths lured the highest percentage of thirteen year olds (32 percent). The distractor one-fourth lured the highest percentage of 17 year olds and adults, 30 and 32 percent respectively. The percentage replying no response or IDK was from 3 percent to 8 percent.

6. RA 11: Which of the following equals 47/5? (Age nine, thirteen, seventeen)

   4 7/5

   9 2/5

   47 1/5

   47 ÷ 1/5

   IDK

   NR

   Results: Only 7 percent of the nine year olds gave the correct answer compared to 65 percent of the thirteen year olds and

81 percent of the seventeen year olds. In addition, 21 percent of the nine year olds responded IDK compared to 6 percent for the thirteen year olds and 4 percent for the seventeen year olds. The greatest distractor for the nine year olds was: 47 ÷ 1/5; 33 percent of them indicated that as the correct answer.

7. RB 10: Which fraction is the GREATEST? (Age nine, thirteen, seventeen)

   2/3

   3/4

   4/5

   5/8

   IDK

   NR

   Results: Three percent of the nine year olds gave the correct response as compared to 26 percent of the thirteen year olds and 49 percent of the seventeen year olds. Seventy-one percent of the nine year olds answered 5/8; 45 percent of the thirteen year olds answered 2/3, and 32 percent of the seventeen year olds answered 2/3.

8. Unreleased exercise: A small set of blue and white marbles was pictured. The respondent was asked what fractional part of the marbles was blue. (Age nine, thirteen)

   All age groups were given an exercise where they were to write in order from the smallest to the highest six common fractions less than one. Discrete elements were used in this question, making it quite different from the shaded parts and breaking the candy bar problems. Results:

a. nine year olds:

21 percent failed to respond or replied IDK
9 percent gave the correct answer
36 percent gave the number of blue marbles.

b. thirteen year olds:

4 percent no response: IDK
65 percent correct
12 percent named the ratio of blue to white marbles

The thirteen year olds showed a knowledge of fraction concepts.

9. Exercise in writing six common fractions in order from the smallest to the largest.

	9s	13s	17s	adults
Correct response	.2%	.3%	15%	20%

Computational facility was checked with two problems, one addition of fractions with unlike denominators and one multiplication of fractions with unlike denominators. The questions were given to the thirteen and seventeen year olds. Only 42 percent of the thirteen year olds responded correctly compared to 66 percent of the seventeen year olds. The problem was: 1/2 + 1/3 = ? Thirty percent of the thirteen year olds responded with 2/5. The results of the multiplication problem were satisfactory: 62 percent of the thirteen year olds and

74 percent of the seventeen year olds replied correctly to 1/2 X 1/4 = . The results of the addition problem indicate a lack of understanding of fractions and quantities and severe failure to develop computation skill or knowledge of computational algorithms.

Decimals: The results of the questions containing decimals were as follows:

1. The nine year olds had a difficult time with placement of the decimal.
2. The thirteen year olds erred when determining the smallest number.
3. Two questions asking for translation from one form of notation to another provided disturbing results:
    a. Age thirteen, seventeen, adults: .3333... represents ?

	13s	17s	Adults
Correct response	12%	41%	52%
No response	30%	17%	29%

    b. Age thirteen, seventeen: 1/5 equals what percent?

	13s	17s
Correct response	43%	65%
No response, IDK	13%	7%
Other incorrect	25%	15%

Comments: Some of the results might be misleading, especially in terms of the nine year olds, since 25 percent of them were in the third grade. Note of this fact was made with respect to an item where algebraic symbolism was used and the item was asked of the nine year olds. It appears that the concept of fractions is poorly understood by the majority of the students.

RESULTS AND IMPLICATIONS OF THE NAEP MATHEMATICS ASSESSMENT: SECONDARY SCHOOL.

Carpenter, T., Coburn, T., Reys, R., & Wilson, J.
The Mathematics Teacher, 1975, 68, 453-470.

Abstract prepared by Mary Ann Konsin

The number of items relating to fractions was small, therefore only a few objectives concerning fractions, fraction concepts, and operations with fractions are discussed.

Nine year olds: The items assessed objectives on intuitive fraction concepts and on ways of using fractions to describe a figure or a physical situation.

Thirteen year olds: Items designed to assess objectives on fraction concepts and operations with fractions.

Thirteen year olds have learned about fraction concepts, although not very well. Below are listed exercises and correct response percentages.

1. Naming fractional parts	65% simple exercise 20% complex exercise
2. Finding a fraction between two other common fractions	55%
3. Selecting a fraction closest to 3/16 from a given list of fractions	19%
4. Changing an improper fraction to mixed form	65%

5. Adding two unit fractions	42%
6. Multiplying two unit fractions	62%

Some of these items were administered to the seventeen year olds, who usually performed about 20 percentage points higher than the thirteen year olds. Some of the adults were given these items also. They performed between the two other groups, except for one exercise. Their results were closer to the thirteen year olds than they were to the seventeen year olds.

The low performance is attributed to the intrinsic difficulty with the concept of fraction rather than to a lack of attention to fractions and fraction concepts in the upper elementary grades. It appears that thirteen year olds see fractions parts, equivalent fractions, and computational algorithms as separate, unrelated topics. These concepts need coordination based on solid intuitive development and understanding.

Comments: The tone of all of these articles seems to be that the intuitive work with fractions has been neglected in the early grades, causing problems when computational skills are developed.

NOTES FROM NATIONAL ASSESSMENT: ADDITION AND MULTIPLICATION WITH FRACTIONS. Carpenter, T., Coburn, T., Reys, R., & Wilson, J. The Arithmetic Teacher, 1976, 23, 137-142.

Abstract prepared by Mary Ann Konsin

These authors discussed two exercises dealing with addition and multiplication of fractions from NAEP and reviewed some research noting Choate (1975), regarding initial concept work in terms of concrete objects prior to any formal work with the addition algorithm. The students should be able to work with and understand:

1. the ability to identify the unit in various representations of fractions (i.e., reconstruct the unit)
2. idea of part-whole relationship
3. comparing and ordering fractions
4. equivalent fractions
5. renaming improper and mixed forms
6. vocabulary and symbolism

Difficulties with common denominators and equivalent fractions suggest more attention should be given in the fourth and fifth grades to:

a. factors
b. multiples
c. common multiples
d. meaningful instruction regarding the renaming generalization

Comments: The main stress here seems to be on the concept of fraction and the presentation of that concept in a meaningful setting.

FOUNDATIONAL WORK

1. The Child's Conception of Geometry. By J. Piaget, B. Inhelder, and A. Szeminska.

2. Determinants of Part-Whole Perception in Children. By D. Elkind, R. Anagnostopoulou, and S. Malone.

3. Studies in Perceptual Development: II. Part-Whole Perception. By D. Elkind, R. R. Koegler, and E. Go.

4. Comparison of Ratios and the Chance Concept in Children. By E. Fischbein, I. Pampu, and I. Manzat.

5. The Relationship of Area Conservation to Area Measurement as Affected by Sequence of Presentation of Piagetian Area Tasks to Boys and Girls in Grades One through Three. By Thalia Taloumis.

6. The Relationship of Area Measurement and Learning Initial Fraction Concepts by Children in Grades Three and Four. By Douglas T. Owens.

THE CHILD'S CONCEPTION OF GEOMETRY. (Chapter 12. Subdivision of areas and the concept of fractions). Piaget, J., Inhelder, B., & Szeminska, A. New York: Basic, 1960.

Abstract prepared by James Hiebert

Purpose: The central purpose of this work is to trace the spontaneous development of the fraction concept in children. This is assumed by the authors to include the general notion of part-whole relations, and the specific matter of the quantification of parts.

Rationale: This investigation represents part of a larger effort to examine the development of conservation and measurement concepts, and their pyschological interrelationships. Of particular interest is the part-whole relationship and the associated notions of the conservation of wholes, the subdivision of wholes, the quantification of parts, and the part-whole relations as seen in the logical nesting of classes. The concept of fraction provides an appropriate arena for investigating the development of these phenomena.

Research Design and Procedure: As with most of Piaget's work in cognitive development, the basic design is a cross-sectional one, with data gathered in clinical interview situations. Each of the children, ages three through nine, is asked to divide a rectangular or round "cake" into a specified number of equal pieces. They are then asked if the sum of the pieces is equal to the original whole. The responses are used to infer the developmental progression in the acquisition of the concepts, and their psychological interrelationships.

Information concerning sample size and background of subjects is notably lacking. Data are not quantified and not submitted to statistical treatment. In keeping with the intent of the methode clinique, heavy emphasis is placed on the explanations given by the children, rather than on a numerical score of correct responses.

Findings: Children were found to acquire the concept of fractions in several stages. Before age four, or four and one-half, children were unable to divide a whole into halves. They either divided off two fragments of the whole or partitioned it into two unequal pieces. Between ages four and five children could dichotomize a small, regularly shaped whole. However they attempted (unsuccessfully) to trisect a whole by repeated dichotomies, and they did not yet conserve the whole. By ages five and six children solved the trichotomy problem by trial and error, and recognized intuitively that a whole is conserved when subdivided. Children between ages six and seven could divide a whole into thirds immediately and possessed an operational understanding of conservation. Finally, after ages seven or eight, children were able to extend their success with thirds to fractions of fifths and sixths.

Interpretations: As suggested earlier, the authors view the concept of fraction as the synthesis of two primary notions: part-whole relations which are intensive and logical; and part-part relations which are extensive or metric. From their findings they derive seven specific characteristics of these two ideas. They suggest that a complete understanding of fraction requires a realization that: (1) the whole is divisible, potentially composed of separable elements; (2) fraction implies a determinate number of parts; (3) the subdivision must be exhaustive, i.e., there can be no remainder; (4) there is a fixed relation between the number of parts and the number of divisions; (5) the parts have a nesting or heriarchical character, a part is subsumed by the whole, but also is a whole in its own right, susceptible of further subdivision; (6) the whole is conserved under subdivision; and (7) all parts must be equal.

These concepts are suggested to be logically and psychologically interrelated, and are found to be acquired in several stages as discussed previously. Several mechanisms are postulated to account for this development. The most basic of these is an anticipatory operational scheme which arises from the coordination of action. The spontaneous development of this scheme results in the ability to recognize beforehand the part-whole and part-part relationships which are required in the immediate problem. This anticipatory scheme is first operational in situations with small, regular shapes requiring only dichotomy (halves and fourths). It is only gradually transferred to thirds, fifths, and sixths, with circular shapes being more difficult than rectangular or square ones. Trichotomy is thought to be significantly more difficult than dichotomy because it requires anticipating and holding in mind the desired part-size until two separate divisions have been made, whereas dichotomy involves only the comparison of two immediately equal pieces. "Dichotomy is the natural form of logical subdivision [p. 316]."

A second underlying mechanism of development is that of conservation of the whole--the identity and quantity of the whole is preserved under subdivision. Since conservation is a characteristic of concrete operations, a full understanding of fraction is not acquired prior to this developmental stage. However it is noted that children are able to recognize that the pieces can be reassembled into the original whole before they understand that this reconstruction implies a quantitative equivalence of parts and whole. The authors interpret this precursory understanding as an intuitive insight into the empirical possibility of returning to the starting point, a pre-operational understanding of qualitative identity. This primitive notion is not operational, since operational conservation exists only when the subdivision follows an anticipatory scheme which can be retraced mentally.

Operational conservation is claimed to manifest itself concurrently in concrete situations dealing with subdivision of continuous wholes and with hierarchical nesting of classes. However the latter is usually assessed in an abstract, verbal situation which requires restructuring of the operational scheme on a verbal level and consequently appears somewhat later.

Abstractor's Note: This investigation has clearly demonstrated that the fraction concept is not a simple, quickly acquired concept, but is rather a complex concept comprised of several part-whole and part-part notions, and is achieved gradually with many unexpected difficulties along the way. The analysis of the concept into its psychological and developmental components is a major contribution of this work. However, the findings and interpretations of the authors must be considered with the following points in mind.

First, Piaget and associates deal with only one interpretation of fraction in this investigation--the part-whole interpretation represented by the subdivision of area. Whether other interpretations would yield different results is an open question.

Second, all tasks employed in the interviews required children to actually subdivide an area into specified fraction parts. The heavy emphasis on anticipatory schemes as the crucial developmental mechanism would seem to depend on requiring this physical manipulation. Is it possible that different results would emerge in situations where children were only required to judge previously subdivided regions, as occurs in many classroom textbook lessons?

The use of anticipatory schemes in the description of the developmental process also seems to introduce some theoretical problems. Other investigators, as well as Piaget, have shown that children are able to accurately predict the outcome of a situation before they can successfully carry out the task to achieve this result. However, in this investigation the authors dismiss the early ability to predict conservation as a pre-operational intuition. It is not clear how one distinguishes an operational anticipatory scheme from a pre-operational intuition. Furthermore, anticipatory schemes are claimed to emerge from the coordination of action, rather than from precocious thought.

This claim needs further explication since it is difficult to imagine how action could lead to anticipation, when it seems that prediction precedes successful action in many situations.

Finally, the relationship of the logical addition of classes (which in one sense reflects the part-whole concept in the discrete case) to the subdivision of regions needs further investigation. Class inclusion tasks are frequently considered to be among the more difficult of the concrete operations tasks, and are usually performed successfully much later than tasks requiring a part-whole notion in an area or region context. It is not clear whether different operational schemes are required in these different contexts, or whether class inclusion tasks just represent relatively more difficult situations for children.

DETERMINANTS OF PART-WHOLE PERCEPTION IN CHILDREN. Elkind, D., Anagnostopoulou, R., & Malone, S. Child Development, 1970, 41, 391-397.

Abstract prepared by James Hiebert

Purpose: This study was concerned with identifying some of the components in the development of part-whole perceptions in children. Three questions, derived from Piagetian theory and from past research, were of particular interest and provided the purpose for the study. First, do young children's failures in part-whole perceptual tasks result from their inability to "see" the parts and whole in combination, or rather from difficulty in conceptualizing and/or verbalizing what they perceive? Second, is the use of a verbal formula sufficient to mediate both the perception and the reporting of a part-whole configuration? And finally, what is the relationship between children's ability to reconstruct perceptual configurations from memory, their ability to spontaneously use verbal formulas, and their understanding of logical class addition concepts?

Rationale: Piagetian theory suggests that the development of part-whole perception reflects a progressive elaboration of quasi-logical perceptual regulations. Young children who lack these regulations see only the parts in a part-whole configuration. As the regulating mechanisms begin to develop, children are able to attend to the whole, but lose the parts in the process. Finally, at age seven or eight, the regulations are well developed and children can perceptually integrate the parts and the whole.

Previous research by the senior author had suggested that the development of part-whole perceptions are in accord with this theoretical position. However several questions remained unanswered. It is these questions which guide the present investigation, and which serve to isolate some of the components in part-whole perceptual development.

Research Design and Procedure: On the basis of performance on the Picture Integration Test, 40 first-grade children were selected for the study and were equally partitioned into a High Combination (HC) group and a Low Combination (LC) group. Children in the LC group tended to report only parts from a part-whole configuration, while HC children reported primarily wholes and part-whole combinations.

All children were then tested for their ability to reconstruct part-whole configurations from memory, for their verbal descriptions of simple objects made of various materials, and for their understanding of logical class addition in verbal and concrete situations. The latter tests were similar to Piagetian class inclusion tasks.

Data were analyzed using simple *t* tests to compare performance of the LC and the HC groups on each of the three tests.

Findings: Significant mean differences were found favoring the HC group on the reconstruction and verbal description tests (.01 level), and the logical addition test (.05 level).

Interpretations: The conclusions drawn by the authors are formulated in terms of the original three questions. First, since children who did not verbalize part-whole combinations (i.e., LC group) were also those who were least able to reconstruct a perceptual configuration from memory, it appears that the organization of part-whole combinations is not given in immediate sensory experience but is mediated in some way. The mechanisms which mediate the report of part-whole combinations also mediate the reconstruction of these combinations from memory. Therefore, it appears that perceptual regulations are necessary both to "see" and to report part-whole relationships.

Second, the LC group was not able to utilize a verbal formula in the verbal description situations, and consequently little can be said about the sufficiency of a verbal formula in mediating the perception and reporting of part-whole relationships. Third, since HC children performed significantly better than LC children on all three tests, it appears that perceptual regulations underlie the development of the perception, reconstruction, and verbalization of part-whole combinations.

Abstractor's Note: The development of the ability to perceive and verbalize part-whole relationships is central to the construction of the fraction concept. As an attempt to explicate this development the present study represents a useful contribution to the limited body of research investigating the fraction concept from a developmental perspective. However several comments about the study are in order.

The conclusions proposed by the authors concerning the ubiquitous mediating role of perceptual regulators can only be regarded as tenuous inferences. The inclusion of only a narrow age range of subjects and the use of group means for data analysis preclude the possibility of examining the developmental interrelationships of various perceptual abilities, much less the underlying mechanisms of development. One can only conclude that children who perform well on one test (the pretest or the other three) tend to perform well on the others.

The author's selection of dependent measures is quite interesting since the reconstruction test is closely tied to perception, while the logical addition test is a classification task postulated by Piaget to indicate concrete operational thought. On theoretical grounds it would be difficult to argue that they are mediated by the same perceptual regulators. Furthermore, considering the age norms in question, children are usually assumed to understand the part-whole fraction concept before they can successfully respond to logical addition questions. Further investigation is required to identify the role of this logical addition ability--both in its relationship to perceptual abilities and to the construction of the fraction concept.

STUDIES IN PERCEPTUAL DEVELOPMENT: II. PART-WHOLE PERCEPTION.
Elkind, D., Koegler, R. R., & Go, E. Child Development, 1964, 35, 81-90.

Abstract prepared by James Hiebert

Purpose: This study was concerned with the development of children's ability to perceive parts and wholes in a visual configuration in which both parts and wholes have independent meanings. Three questions guided the investigations: (1) Is there an increase with age in the ability of children to perceive both parts and wholes? (2) What is the developmental sequence in part versus whole perception? and, (3) At what age will both parts and wholes be perceived by the majority (75 percent) of children?

Rationale: Piaget's theory of perception suggests that young children's perception progresses from an initial centration dominated by Gestalt-like principles to increasing decentration governed by logical principles. This decentering process should be observable in the perception of part-whole relationships.

Several studies have investigated part-whole perception, but only a few have used figures in which parts and wholes had different and independent meanings. The latter studies indicate that stimulus variables such as proximity and closure are important in determining the primacy of part or whole perception. The present study attempts to explore the development of part-whole perception and the role of these stimulus variables in this development.

Research Design and Procedure: The Picture Integration Test, comprised of a series of pictures (e.g., animals, faces) made from common objects (e.g., fruits, vegetables), was administered individually to 195 children from four to nine years of age. The experimenter asked the children to describe what they saw in the picture, and recorded their responses as either part, whole, or both part and whole responses. All children had been previously classified as average or above average intelligence based on the mean I.Q. score of their school sample, and as belonging to a low or high socioeconomic group based on the geographic location of their school.

Percentages were computed for part, whole, and part-whole responses for the five age levels and the two I.Q. groups. Chi-square tests were run to check for significant differences resulting from socioeconomic, intellectual, and sex status.

Findings: No significant differences due to socioeconomic, intellectual, or sex status were found with respect to the age at which part and whole were perceived nor with respect to the developmental order of part and whole responses.

Regarding the original three questions posed in this study, it was found that: (1) there was a regular increase with age in the percentage of children who perceived both parts and wholes, (2) children of all ages perceived parts more readily than wholes, and (3) by nine years of age a majority (78.6 percent) of children were able to perceive both parts and wholes.

Interpretations: The earlier appearance of part perception was attributed to the configural character of the stimulus materials. In the Picture Integration Test the parts are more closed and have a more recognizable form than the wholes. The authors conclude that a statement about the developmental sequence of part versus whole perception must take into account the character of the stimulus materials.

The findings of a regular increase with age in integrated part-whole perception is suggested to be in accord with Piaget's decentering theory. Furthermore, the authors draw an analogy between part-whole perceptual development and multiple classification conceptual development. As in multiple classification, the child must be able to attribute two percepts (concepts) to the same form. For example, the child must perceive the banana (part) to be at the same time a leg of an animal (whole). Stages in part-whole perceptual development, analogous to those in the development of multiple classification, are described. These stages proceed from complete centration to complete decentration and integration.

Abstractor's Note: Viewed from a developmental perspective, the perception of part-whole relationships is a primitive but essential part of any complete description of the fraction concept. The ability to perceive part-whole relationships would seem to be a necessary, although not a sufficient, condition for acquiring a concept of fraction. As such, this perception represents a prerequisite notion in the development of this concept.

It is not yet as clear what level of perceptual proficiency is required for a conceptual understanding of "fraction." If the ability to perceive part-whole relationships as measured in this study is a prerequisite for conceptualizing fractions, then the age-related findings would have serious implications for grade-placement of fraction concepts. Further investigation is needed to determine the relationship between the perception of part-whole relations and the conception of these relations as embodied in fractions.

The authors' attempt to examine individual differences in perceptual development related to socioeconomic or intellectual status is inadequate. The classification of children into higher or lower groups based on the mean level of the school sample to which they belong permits inferences about groups (with certain characteristics) but not about individuals.

COMPARISON OF RATIOS AND THE CHANCE CONCEPT IN CHILDREN. Fischbein, E., Pampu, I., & Manzat, I. Child Development, 1970, 41, 377-389.

Abstract prepared by James Hiebert

Purpose: This study sought to investigate the development in children of the ability to respond accurately in probability situations by comparing numerical ratios, and the effect of prior systematic instruction on this ability.

Rationale: The authors cite previous work by Piaget and others which had shown that appropriate probabilistic judgments could be made by children as young as five years of age. Furthermore, the grasp of the notions of ratio, proportion, and probability had been shown to increase with age through the formal operations period. However most of these experiments were limited to the investigation of spontaneous behavior.

The present authors wished to extend these findings by considering the effects of systematic instructions, as well as the effects of several stimulus variables, such as the size and type of the ratio involved in the probability situation.

Research Design and Procedure: The subjects were 180 preschool, third grade, and sixth grade Hungarian children. Each child was presented with 18 problems requiring a judgment as to which of two jars, both containing black and white marbles, would yield the best chance of getting a black (white) marble when selecting one. The type of ratios of black to white marbles were varied to yield three categories of six problems each. One-half of each of these types of problems utilized small numbers, while the other half employed numbers exactly twice as large. Responses were scored as correct or incorrect.

Prior to administering these criterion tasks, each age group of 60 children were subdivided into three groups of 20, with each subgroup randomly assigned to one of three instruction procedures. The instructional treatments, individually administered, varied from simple illustration of the correct response to direct instruction of a reducing or expanding technique to aid in comparing the ratios. Instruction was completed in a single session of 30 to 60 minutes.

The basic design was a 3 x 3 x 3 factorial design--three age levels, three instructional conditions, and three categories of problems. Data was analyzed using analysis of variance, with repeated measures on the last factor.

Findings: Significant main effects were found for age (older children were more proficient than younger children), instruction condition (direct instruction more efficient than illustration), and problem type. Significant first-order interaction effects were found for age x problem type (the type of problem involving equivalent ratios produced the most substantial changes with age) and instruction x problem type (the type of problem with the same numerator, or denominator, in both ratios showed little difference with instruction due to ceiling effects). The differences between the small and large number problems were nonsignificant in most cases.

Interpretations: Third grade children were found to benefit most from instruction. It is suggested that initially their responses were similar to those of the preschool children, but with brief instruction they were able to adopt a mode of reasoning usually absent until the period of formal operations several years later. This finding, which appears to contradict Piagetian stage theory, is explained by the authors in two ways. First, conventional Piagetian theory is based primarily on spontaneous responses. This study examined the effects of instruction. Second, these results may indicate that difference between stages become less pronounced at the higher stages. Consequently, children at the concrete stage can be trained in formal operations with greater ease than those at the preoperational level could be trained in concrete operations.

Abstractor's Note: In this study the authors have demonstrated that children as young as nine to ten years of age may be taught to compare and order two ratios. Since the instructional procedures were not elaborate or extensive it appears that third grade children are able to deal with these concepts, at least in the limited sense investigated here, with a minimum of instruction.

The authors' conclusions concerning the theoretical implications of the results must, however, be considered within the following constraints. Piaget has suggested three criteria which should be used in evaluating training studies such as this one: (1) At what level of development were the children initially? (2) To what extent did the newly acquired skill transfer to other related tasks? (3) How well was this new skill retained over time? The present study failed to employ measures which would enable the authors to answer any of these questions.

Training of a particular skill, even a concrete operation with preoperational children, which the authors suggest to be more difficult, has been frequently demonstrated. The criteria mentioned above point out that it is not the training of a specific skill in a limited context which is theoretically significant, but the amount of generalizability, stability, and structural reorganization which is evidenced. The results of this study do not address any of these issues.

Finally, the relative lack of instructional effect exhibited by the preschool and sixth grade subjects appears to be more a matter of task selection than instruction receptiveness. The preschool children apparently did not understand the problems and showed continual random responses, while the sixth graders were successful at the outset and could not improve through instruction due to ceiling effects. Therefore, given the tasks used in this study, the third grade subjects were the only ones who could benefit from instruction.

THE RELATIONSHIP OF AREA CONSERVATION TO AREA MEASUREMENT AS AFFECTED BY SEQUENCE OF PRESENTATION OF PIAGETIAN AREA TASKS TO BOYS AND GIRLS IN GRADES ONE THROUGH THREE. Taloumis, T. Journal for Research in Mathematics Education, November 1975, 6(4).

Abstract prepared by Maurice Burke

Purpose: The purpose of this study was to measure the relationship between the two concepts, area conservation and area measurement, and to consider its implication for placing area measurement in early grades. The questions addressed were (1) Do pupils in grades one through three who scored higher on area measurement also score higher in area conservation? (2) Would the concept of conservation of area have to be attained to facilitate that of area measurement?

Procedure: Three area conservation tasks and two area measurement tasks, all representative models of area tasks reported by Piaget, were individually administered to each of 168 middle class children ages six years five months to nine years four months. They were administered in two sequences, one putting the conservation tasks first and one putting the measurement tasks first. Statistical procedures included an analysis of variance, Pearson Product Moment correlations, and multiple regression correlations for subgroups.

Results: The findings indicated that (1) scores were higher on whichever group of area tasks (conservation or measurement) came second; (2) there was a significant correlation (.66) between area conservation scores and area measurement scores; (3) boys and girls did not score significantly differently on area conservation tasks and area measurement tasks; and (4) scores in area conservation tasks and area measurement tasks showed no significant differences in amount of increase from grade to grade.

Implications:

(1) Training in area measurement may improve a child's performance in area conservation.

(2) Thus Piaget's theory that the ability to measure quantities is dependent on acquired concepts of conservation does not seem to be tenable.

Abstractor's Note: This article highlights some of the problems involved in the question of readiness with fractions.

THE RELATIONSHIP OF AREA MEASUREMENT AND LEARNING INITIAL FRACTION CONCEPTS BY CHILDREN IN GRADES THREE AND FOUR. Owens, D. T.
A paper presented at the 55th Annual Meeting of the NCTM, 20-23 April 1977, Cincinnati, Ohio.

Abstract prepared by Maurice Burke

Rationale: The Area Model is commonly used to introduce fractions. However, some children in grades three and four do not have well developed area concepts. This study addresses itself to the following questions: (1) What effect does a child's area concept have on his or her ability to learn fraction concepts using Area Models? and (2) What effect does grade level have on the ability of the children to learn fraction concepts at the third and fourth grade levels?

Design: One hundred and one third and fourth grade students were administered an Area Concept Test consisting of six items--two of which dealt with conservation of area and four dealt with measurements. From the low achievers (0-2 correct) and the high achievers (4-6) 56 pupils were randomly selected for instruction. Two instructional groups were formed mixing third and fourth grade, high and low level children. Instruction was based on a revision of the Initial Fraction Sequence (Muangnapoe, 1975) and was administered in seven 40-minute periods. Emphasis was placed on the Area Model (omitting Number Line and Set Models). Paper folding activities were widely used. On the eighth day an achievement test, the Fraction Concept Test (FCT), was administered. This test is similar in nature to the worksheets completed by the students. Along with the items directly related to fractions and fractional symbolism were included items on (1) identifying figures which have equal-sized parts and (2) deciding whether two diagrams show equal areas shaded. A transfer test was administered consisting of eight items on equivalence, mixed numerals, and the "part of a set" meaning of a fraction.

Results: Age and grade had no relationship to either area measurement or fraction learning. Only the correlation of .50 between ACT and FCT was significant. This reflects the results of the analysis of variance which shows the high level area students scoring significantly better on both the FCT and the transfer test than the low level area students.

Group Means for Fraction Concept Test and Transfer Test

	Low Level	High Level	Total	# of Items
Fraction	35.0	42.3	38.6	51
% Means	69%	82%		
Transfer	1.5	3.3	2.4	8

Conclusions:

1. It appears that the children in grades three and four are equally capable of informal area measurement and learning initial fraction concepts under the Area Model.
2. Area concept does have an effect on fraction learning based on an area model.

3. Even low level area concept children can learn the fraction concepts as indicated by their mean of 70 percent on the achievement test.

Abstractor's Note: Without looking at the FCT one must be careful when reading these results. Since what amounted to area measurement type tasks occurred in the FCT, differences in the achievement scores may not reflect differences in the learnings that took place in the instruction, but may only reflect the differences that were already detected between the two groups by the Area Concept Test.

FURTHER STUDIES ON READINESS AND CONCEPTS HELD BY CHILDREN

1. Fraction Concepts Held by Young Children. By A. Gunderson and E. Gunderson.

2. Assessment of Mathematical Concepts of Five Year Old Children. By Anthony N. Schwartz.

3. Mathematics in Kindergarten. By George W. Schlinsog

FRACTION CONCEPTS HELD BY YOUNG CHILDREN. Gunderson, A., & Gunderson, E. *The Arithmetic Teacher*, 1957, *4*, 168-173.

Abstract prepared by William H. Kraus

Purpose: To find out what concepts and ideas young children have about fractions.

Rationale: Since teachers frequently report difficulties in teaching fractions and operations with fractions in the upper elementary grades, the researchers were interested in whether students in the lower elementary grades had any knowledge of fractions. If the children did have some sort of fraction concept, it was conjectured that this knowledge base could be built on in such a way that in later grades the children would not have as much difficulty learning other fraction ideas.

Method: Twenty-two children from Willett's Road School in East Williston, Long Island, New York were individually interviewed. The interviews did not follow a standardized format. Questions were asked to determine how much the child knew about fractions. A flannel board and circles (pies) were used to illustrate the questions. Two short introductory lessons dealing with halves, fourths, eighths, thirds, and sixths were taught prior to the interview. The children had had no other formal experience with fractions.

Conclusions: The researchers concluded that the concept of fractions could be introduced in second grade, since all the subjects demonstrated some ability to grasp the idea of a fraction. Most were able to identify fractional parts, compare fractions, and to generalize that the more parts a whole is divided into, the smaller the parts become. The researchers stress that the above results were achieved with manipulatives without any written representation for fractions and that similar results should not be expected with a formalized symbolic treatment of the same ideas.

Abstractor's Note: Although over 20 years old, this study deals with instructional ideas that are certainly not outmoded. And although the rather loose procedures followed in the research do not provide us with any "hard facts" about what seven year olds know or do not know about fractions, the study does provide indications that many seven year olds may be able to learn a considerable amount about fractions on a non-symbolic level.

ASSESSMENT OF MATHEMATICAL CONCEPTS OF FIVE YEAR OLD CHILDREN.

Schwartz, A. N. Journal of Experimental Education, Spring 1969, 37.

Abstract prepared by Maurice Burke

An instrument (the Schwartz Early Math Inventory) was devised to assess the mathematical achievement of children ages three through five. A realiability of .94 was obtained on the 81 items administered to 215 kindergarteners.

Results on fractional recognition are as follows (questions were read to the children):

"Mark the jar that is 1/2 full of water"	41%
"Mark the frame that shows 1/4 or one-quarter pie"	35%
"Mark the frame that shows 1/3 of a pie"	24%
"Mark the jar that is 1/4 full of water"	12%

MATHEMATICS IN KINDERGARTEN. Schlinsog, G. W. The Arithmetic Teacher. April 1962.

Abstract prepared by Maurice Burke

In recommending a mathematics curriculum for the kindergarten, the author suggests that kindergarten children be introduced to halves, thirds, and fourths, as well as qualitative measurement activities.

A NEW RATIONALE FOR THE TEACHING OF FRACTIONS. Reiss, A. P.
Harvard Educational Review, Spring 1955, 105-115.

Abstract prepared by Maurice Burke

In this paper the author explores the original development of the concept of fractions for suggestions in regard to teaching. She says that children would not gain anything by being told about the history of fractions, but, if applied to the child's own constructive activities, historical insights can become a further source for sound suggestions for a rejuvenation of conventional teaching.

To the child, no less than to his ancestors before him, fractions are at first the genuine, palpable parts or portions into which a concrete whole has been subdivided. At the most primitive level, one nearly always finds that half and quarter are terms understood in close relation to the act of breaking into parts, either once, or once and once again. Measuring is also a critical activity for releasing in the mind of the child the formative notions which lead toward a mathematical concept of fractions.

Thus, Reiss feels that in the elementary grades children should be given much work with concrete objects and measurement to establish the basic concepts of fractions. Manipulation of concrete objects should precede the use of diagrams and measurements. The set/subset usage of fractions (the notion that a whole number can be the fractional part of another whole number) should not be presented until the intermediate grades. With emphasis on the measurement interpretation

INTERPRETATIONS AND RECOMMENDATIONS

1. A New Rationale for the Teaching of Fractions. By Anita P. Reiss

2. Fractions and Related Symbolism in Elementary School Instruction. By Maurice L. Hartung

3. The Meaning of Fractions. By David Rappaport

4. Goals for School Mathematics: The Report of the Cambridge Conference on School Mathematics: 1963.

5. Fractional Numbers. By A. Coxford and L. Ellerbruch.

6. Paper Folding and Equivalent Fractions--Bridging a Gap. By Harry Bohan.

FRACTIONS AND RELATED SYMBOLISM IN ELEMENTARY SCHOOL INSTRUCTION.
Hartung, M. L. The Elementary School Journal, April 1958, 377-384.

Abstract prepared by Maurice Burke

Discussing the meaning of fractions, the author notes that concrete situations provide a useful stage setting for developing the fraction concept. However, the use of concrete situations does not ensure that understanding will emerge. He goes on to say that fractions give us measurement information showing us the relation between the part and the whole. Thus it is not adequate to say that three-fourths indicates three of four equal parts of some whole. It is necessary to understand the concept of a relation to really understand fractions.

The ratio meaning is not another separate meaning; rather it is an essential aspect of the understanding of fractions. For example with three-fourths, the concept includes not only the idea that the symbol represents three of four equal parts of something but also the relation (ratio) of these three parts to the whole.

He concludes that fractions have essentially one underlying meaning with many uses; and that it would clear the air if discussion on fractions sought to establish one basic concept that includes several related ideas.

of fractions, the child experiences fractions from the angle of multiplication rather than division: unit fractions are dealt with as sub-units, i.e., sub-multiples, of the basic unit. Thus fraction is taken from the realm of a piece of the whole (the child's initial view of fractions) and is treated as a sub-unit, implying a multiplicative relationship to a whole.

GOALS FOR SCHOOL MATHEMATICS: THE REPORT OF THE CAMBRIDGE CONFERENCE ON SCHOOL MATHEMATICS: 1963. Boston: Houghton-Mifflin, 1963.

Abstract prepared by Maurice Burke

(Section 5) Curriculum for Elementary School (K - 2)

The Conference recommended extensive use of the number line for representing the natural numbers, integers, and fractions with small denominators. The number line should also be used to introduce decimals by a change of scale. For applications, they recommended the measuring of lengths, areas, volumes, etc., and use of physical interpretations for one-half, one-third, one-fourth, two-thirds. They also proposed that remainders be represented as fractional parts of the divisors in some cases.

Abstractor's Note: It is not clear just exactly how they hoped to see fractions developed. It is clear that the number line representation and an abstract theoretic presentation of the rationals were part of the recommendation.

THE MEANING OF FRACTIONS. Rappaport, D. School Science and Mathematics, April 1962, 241-244.

Abstract prepared by Maurice Burke

In his discussion of fractions, the author lists the following as different usages of fractions and recommends that all of them be used in the teaching of fractions:

1. A fraction a/b is a symbol for an operation in which a base unit is divided into b equal parts and a of these parts specially designated.
2. A fraction 3/4 represents a quantity, i.e., three-fourths of some whole.
3. A fraction is a symbol that indicates division.
4. A fraction is a symbol for a ratio (a comparison between two numbers).

Second, they identify a hierarchy of ideas that are necessary for a child to learn in order to develop a useful concept of fractions. The hierarchy, which begins with the idea of a unit and extends to such ideas as equivalent fractions and comparing fractions, follows:

1. Units
    - Identifying the number of units
    - Identifying more or less than a unit
2. Parts of a Unit Using Concrete Materials
    - Identifying the number of pieces in a unit
    - Identifying equal-sized pieces
    - Making a unit into equal-sized pieces
3. Oral Names for Parts of Units
    - Establishing fraction names
    - Using fractions to answer "how much"
    - Identifying fractions equal to one
4. Written Fractions for Parts of Units
    - Oral to written
    - Concrete to written
    - Written to oral
    - Written to concrete
5. Representing Fractions with Drawings
    - Transition from objects to diagrams
    - Repetition of parts one to four of the sequence using diagrams

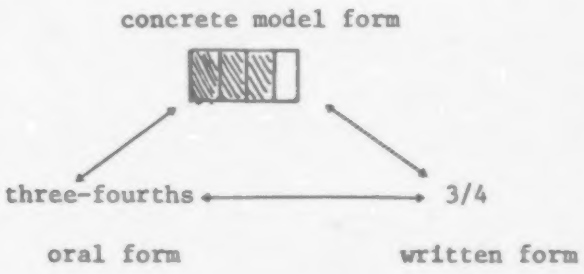
FRACTIONAL NUMBERS. *Coxford, A., & Ellerbruch, L. Mathematics Learning in Early Childhood, 37th Yearbook of the National Council of Teachers of Mathematics, Reston, Virginia: 1975, 191-203.

Abstract prepared by William H. Kraus

Although not strictly speaking a research report, this chapter is being included here for two reasons: (1) it is based, in part, on research being carried out at the University of Michigan (major investigators: Joseph Payne, James Greeno, Stuart Choate, Chatri Muangnapoe, Hazel Williams, Patricia Galloway, and Lawrence Ellerbruch), and (2) familiarity with the systematic approach to the development of the concept of fractions presented in the chapter will be useful to other researchers.

Abstract: The authors' purpose appears to be to present a systematic approach to developing the concept of fractions in young children. The authors identify two important considerations.

First, they stress that it is important that the child have facility in making connections among the concrete model form of a fraction, the oral form of the fraction, and the written form; the diagram below illustrates these connections.



PAPER FOLDING AND EQUIVALENT FRACTIONS--BRIDGING A GAP. Bohan, H.

Arithmetic Teacher, April 1971, 18(4), 245-249.

Abstract prepared by Charles Mitchell

Purpose: The purpose of this paper was to determine if paper folding activities could be used to bridge the gap between conceptual work previously done with equivalent fractions and the generalization

$\frac{a}{b} = \frac{a \cdot n}{b \cdot n}$.

Rationale: The use of concrete models and manipulative devices in the early stages of the development of mathematical ideas has long been emphasized. However, some teachers fail to take students from the concrete representation to the concept behind it. With some models, the processes children use at the concrete and semi-concrete levels are not closely related to the processes they will later use in dealing with abstract symbols. One must ensure that the choice of model will provide transfer of training from the concrete model to the abstract concept that it was chosen to support. Bohan felt that the great majority of modern textbooks he surveyed failed to fit the model to the concept.

Research Design and Procedure: This article was based on Bohan's doctoral thesis completed at the University of Michigan and much of the usual information found under this rubric was not reported by the author in this paper. However, he did mention that there were six classes and three treatments. Because of initial differences in the

6. Extending Notions of Fractions

    Fractions greater than one

    Mixed forms

    Set model--using sets to answer questions such as, "What is 1/2 of 6?"

    Comparing fractions

    Equivalent fractions

The authors present a number of systematically organized activities that fit the theoretical framework they have built up. This serves not only to clarify each part of the framework but to indicate the practical applications of the theory as well.

Abstractor's Note: One might argue with individual pieces of the author's theoretical framework for the development of the concept of fractions. For example, they suggest using ordinal counting--first, second, third, fourth, etc.--to introduce the fraction names, third, fourth, etc. This approach might lead to considerable confusion, since the ordinal idea of "third" and the fraction idea of "third" are quite different. However, the overall framework seems well constructed. The pieces of the framework and the relations between the pieces are testable, and thus this chapter would provide a possible framework for research. (The suggested activities might also be useful to teachers and researchers.) Even if this framework is not adopted (or adapted) by the researcher, familiarity with the ideas would still seem to be worthwhile.

Abstractor's Note: It is not surprising that the "property of one" approach was determined to be the poorest. As described, it was approached symbolically. If this was the case, perhaps this approach should be modified so that it is more concrete.

The author stressed that a good model should provide transfer to the abstract concept, but overlooked the undoing aspect of equivalent fractions. Not only must children get from two-thirds to four-sixths, but vice versa. The reason that the "property of one" approach may prove to be important is that it relates very closely to the process of eliminating common factors.

Bohan's work is very important to the study of fractions. Equivalent fractions and the related material have long been recognized as a problem area.

treatment groups, an analysis of covariance was used with arithmetic achievement as covariate.

The first treatment was called the "property of one" approach and was illustrated by $\frac{a}{b} = \frac{a}{b}\cdot 1 = \frac{a}{b}\cdot\frac{n}{n} = \frac{a\cdot n}{b\cdot n}$. The second treatment, called the "usual" approach, began with the use of shaded diagrams. The size of the figures, and the area shaded in each, was the same. Each figure was divided into a different number of equal parts. The third treatment was the paper folding approach. After answering questions about a figure, the paper was folded along dotted lines. Without unfolding, questions were asked about the number of parts of the paper, the number of shaded parts, the fraction it represented, and its relationship to the previous fraction. Papers were unfolded to check answers and soon children were asked to generalize.

Findings: There were no significant differences between adjusted means on either immediate posttests or retention tests on addition, subtraction, or multiplication. On an immediate posttest of concepts and skills related to equivalent fractions, both the "usual" and paper folding approaches showed a significant superiority over the "property of one" approach. On a retention test given three weeks after instruction ceased, the paper folding approach was superior to the others, which were about the same at that point.

Results: Paper folding promoted better attainment of objectives related to fractions.

Working Paper No. 218
Part 2 of 2 Parts

THE TEACHING OF INITIAL FRACTIONS CONCEPTS

by

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- providing assistance to educators which helps transfer the outcomes of research and development to improved practice in local schools and teacher education institutions

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SUMMARY

Research findings and recommendations listed here can be summarized according to how they relate to the following questions:

Status Question: What concepts of fractions do children possess prior to formal instruction?

Readiness Question: How soon are children able to learn initial fraction concepts?

Representation Question: What representations are easiest or hardest for children?

Sequence Question: What does research suggest about the sequencing of initial fraction concepts in instruction?

Status Question: Campbell (1974) found that very few children aged five, six, or seven could construct or show one-fourth or one-third of a whole. The NAEP (1975) reports that only 20 to 35 percent of nine year olds show knowledge of elementary fraction concepts such as recognizing (1) one of three equal parts of a candy bar as being a third, or (2) what one-fourth of eight objects were. Sebold (1946) found that very few third graders could name the fractional part of a figure that was equally divided and only about one-fourth of the children used the equal parts criteria. She also found that for about 50 percent of the third and fourth graders "fraction" connoted one-half, and any part of a whole was a half or a quarter.

On the other hand, Suydam and Reidesel (1971) report results that show kindergarten children having a good grasp of halves, thirds, and fourths. Schwartz (1969) found that 41 percent of the kindergarten children he tested could "mark the jar that was 1/2 full of water" and that 35 percent could "mark the frame that shows 1/4 or one-quarter pie." However, only 12 percent could mark the jar that was 1/4 full of water, and 24 percent could mark the frame that showed 1/3 of a pie.

Sebold (1966) reported that about 33 percent of the third graders indicated some understanding of fractional parts of groups, and Campbell (1974) reported that most five, six, and seven year olds had a clear understanding of what one-half a whole was (using a concrete object) and a majority of the seven year olds understood the interpretation of one-half of a set when it consisted of four concrete objects.

In general, research shows that prior to formal instruction children's understanding of initial fraction concepts increases with age. "Half" of a whole is understood by a majority of children aged five and older. Less than a third of these children, however, could use thirds and fourths to describe fractional situations (although more could pick out an object or diagram that represented some given fraction, e.g., one-third or one-fourth). Unit fractions seem to be easier than proper fractions and understanding of fractional parts of sets appears to be roughly the same as fractional parts of whole. Prior to formal instruction, the meaning of fractional terms is vague and imprecise to most children. They generally lack the equal parts criterion and commonly fail to involve the number of parts criteria. However, the terms (such as half, third, or fourth) are familiar and generally do connote some kind of breaking up of a whole.

Readiness Question: On the developmental side, Piaget (1960) provides an analysis of part-whole understandings prerequisite to initial fraction concepts. Among the elements of the understanding is that the whole is divisible (potentially composed of several elements) and that the whole is conserved under subdivision. These results suggest that since conservation is characteristic of concrete operations, a full understanding of the initial concepts of fractions under the part-whole interpretation is not acquired prior to this stage.

One could question the accuracy of this last conclusion, however, on the basis of Taloumis' (1975) finding. She questions just how much the ability to measure quantities depends on conservation. Thus, initial fraction concepts, insofar as they rely on the notion of "equal-sized parts," could be learned before the child had reached full conservation. Although Owens (1977) found that the area concept (including conservation) and measurement did have an effect on fraction learning based on the Area Model, even students low in area concept and measurement could learn the initial fraction concepts.

Neverthelss, there is some question about just how ready children are to master the initial fraction concepts. Hartung (1958) and others emphasize seeing the part in its relationship to the whole as fundamental to fractions and most researchers agree on the need for work with equivalence partitionings in initial fraction learning. To this end, Piagetian theory makes two claims: first, children under six have difficulty with equivalence partitionings and part-whole understandings while children from six to eight years of age generally have an operational understanding of conservation and can easily perform equivalence partitionings. Secondly, very often children six and under lack certain perceptual regulations and thus see only the parts in a part-whole configuration, but children at the ages of seven or eight are usually able to perceptually integrate the parts and the whole. (Elkind, 1964, found that by age nine, 78.6 percent of the children tested were able to perceive both parts and whole within a given configuration.)

In general, on the developmental side, it is not yet clear what levels of perceptual proficiency and area conservation are required for a conceptual understanding of fractions. At this point, research indicates that both are important factors with respect to the readiness question but more work needs to be done specifically relating levels of these and related factors to initial fraction learning. Kieren (1996) reports planned research by Sambo and Babcock attempting to examine the relationship between measurement abilities and initial fraction learning.

The question of what children can learn has turned up some results. Schlinsog (1962) recommends that kindergarten children be introduced to halves, thirds, and fourths. Gunderson (1957) found from two short introductory lessons dealing with halves, thirds, fourths, sixths, and eighths, that children in second grade demonstrated an ability to identify fractional parts, compare fractional parts using manipulatives, and generalize that the more parts a whole was divided into the smaller the parts became. In a more extensive study, Sension (1971), found that instruction was effective in second grade on both set and region interpretations. For example, on 5 of 17 item formats, second graders with instruction on fractions scored above 80 percent on the average while those without formal instruction scored below 20 percent.

The Michigan studies (Muangnapoe, 1975; Galloway, 1975) using the Initial Fraction Sequence report that children in second and third grades are capable of learning basic fraction concepts using a region interpretation. Children in those grades have more difficulty with the set and the number line representations, the latter being the hardest. (This agrees in part with the Committee of Seven finding on the set/subset interpretation of fractions.) Children in grades four, five, and six were able to master most of the Initial Fraction Concepts but had some difficulty in working with number line representations. Galloway (1975) reports that children as early as grade one (80 percent) were able to identify a unit, equal-size parts, and fractions less than one. She also reports that there seem to be significant gaps between grades one and two, and between grades three and four in ability to learn initial fraction concepts.

Thus, research seems to indicate that children, at least from second grade on, can learn initial fraction concepts under the part-whole interpretation with the whole being a continuous quantity. These concepts include recognition, representation, and comparison of fractional parts. The number line interpretation seems difficult for the early grades one to three and there is some conflict (Sension vs. the Michigan investigators) about how easy it is for children of these grades to learn and use the set/subset interpretation.

Interpretation Question: Coburn (1974), comparing region to ratio interpretations, found that the region interpretation provided a better basis for developing fractional concepts leading to addition and subtraction of fractions. Muangnapoe (1975) found that the region interpretation was easier for children than the number line model. (The region interpretation ran into trouble only with improper fractions and mixed numbers--a concept difficult for children under all interpretations.) Novillis (1976) concluded similarly that associating fractions with part-whole and part-group situations was prerequisite to associating a fraction with a point on a number line. And the NLSMA data showed number lines to be more difficult than either set or region representations for fourth grade students.

Sension (1971) found that three initial fraction instruction sequences, set, area, and set plus area, were all equally effective for students in grade two; and that, in fact, the combination treatment transferred to the Number Line Model better than the other two. This seems counter to what the Michigan investigators found. Green and Muangnapoe both found the Set Model to be more difficult than the Region Model, pointing to a difficulty with the unit idea. The Michigan studies also seem to show that interference arises in instruction when region and set interpretations are mixed. In developing the Initial Fraction Sequence, the Michigan investigators found that the Set Model was too difficult to be included in the sequence (fourth graders were being used) and they conjectured that children's understanding of the Region Model, with emphasis on measurement, seemed to disintegrate after teaching the Set/Subset Model.

Thus, most questions still remain open on the interpretation question. It does seem to be the case the the Number Line Model is more difficult than either the Region or the Set Models. Research also seems to indicate, although not unanimously, that the Region Model is easier for children than the Set Model, but more work has to be done here before the question can be settled. Commentators generally agree that children should be exposed to fractions under many different representations, including set, region, and number line. Yet research says very little about how this should be done. Research does, in fact, lend some plausibility to the interference hypothsis, thus further complicating the situation.

Sequence Question: On this question research comes up with many diverse results. A consistent conclusion is that most children do not have an adequate experiential background for a good understanding of fractional terms (Sebold, Kieren, etc.). Most investigators recommend extensive use of concrete objects, diagrams, and other manipulatives in teaching initial fraction sequences. (Suydam and Weaver, 1976, p.S-1). Reiss (1955) recommends much use of concrete objects before the use of diagrams, and Campbell (1974) found that the use of concrete objects was easier than the use of diagrams for five, six, and seven year olds. Muangnapoe and Galloway both embodied this same conclusion in the way they taught the Initial Fraction Sequence.

Galloway (1975) found that children in grades one to three had difficulty in writing and diagramming or otherwise representing fractions. Muangnapoe (1975) found that children had more difficulty in writing a fraction that would describe a given diagram than in determining whether a given fraction described a given diagram.

Research (Sebold, Muangnapoe, and others) shows that pupils have difficulty realizing the need for equal-sized parts. To remedy this Muangnapoe recommended that examples of whole divided into non-equal parts should be provided when children are learning to recognize fractional parts. Novillis (1976) extends this to the notion of congruence and asserts that associating a fraction with a situation involving congruent parts was prerequisite to associating a fraction with a situation involving non-congruent (but equal) parts.

As far as types of fractions are concerned, Sebold (1946) found that the concept that a fraction could be more than one of the equal parts of a whole was not difficult for children once they had learned the concept of a unit fraction. This suggests that counting fractional parts is a slightly harder concept than that of the unit fraction. Sebold also found that mixed numbers seemed easier to use than improper fractions. She felt that one of the problems with improper fractions was identifying the unit when many units were present. (Muangnapoe, Kieren, and Novillis all recognize a similar problem existing here.) Less than one-fourth of the fifth, sixth, and seventh graders could associate an improper fraction with a diagram while over 50 percent were able to associate a mixed number with a diagram. Muangnapoe found fractions greater than one difficult and recommended that improper fractions be introduced after mixed numbers as alternate ways of representing mixed numbers.

In line with the problem of the unit, Kieren and Novillis claim that up to age twelve or thirteen children have great difficulty dissociating the notion of a unit from "everything there." Recognizing this problem and aware of the importance of the unit concept in initial fraction developments under the part-whole interpretation, Coxford and Ellerbruch (1975) place much emphasis on the unit throughout their recommended sequence or fractions.

Finally, it can be conjectured, with some support from research, that the Region, Set, and Number Line Models should be introduced in that order in initial fraction training. In any case, research does question the Cambridge Conferences recommendation of extensive use of the number line representation in the early grades.

Though we have reported many results from research it seems fair to say that many of the primary questions have been left unanswered. Even though initial ground has been broken in all of the areas we listed, the questions of readiness, status, sequence, and representations are still open. There is still a need for concept analysis outlining the major understandings a child must have to learn fractions under the part-whole interpretations. Components of this concept analysis must be examined with respect to (1) the questions of status: What portions of the conceptual framework do children commonly attain and what misconceptions do they generally hold relating to those sections; (2) questions of readiness and sequencing: When are children ready to master the various components and is there an optimal sequence through which to proceed in developing the concepts; and (3) questions of representations and how children's concepts develop with respect to those representations. These are just several of the directions that research needs to go with respect to the development of initial fraction concepts under the part-whole interpretation. It was within the context of these considerations and needs that the pilot studies reported in the fifth chapter were formulated.

IV

INITIAL FRACTIONS CONCEPTS IN SELECTED TEXTBOOKS

INTRODUCTION

As part of the review of the literature about initial fraction concepts, treatments of these concepts by several well known authors of textbooks were reviewed. The texts chosen cover two decades.

The School Mathematics Study Group (SMSG) textbooks were reviewed because these books represented the work of many famous educators, who were attempting to write the definitive "new mathematics" book during the 1960s. This series of texts was written first for higher levels of school mathematics, and was then progressively extended down to the primary grades.

The next text chosen was the 1966 edition of the Holt, Rinehart and Winston series: Elementary Mathematics: Patterns and Structures. This book was chosen to represent a popular text of the period.

Several texts from the early 1970s were then reviewed: Harcourt Brace Jovanovich (1972), Harbrace Mathematics; Addison-Wesley (1973), Investigating School Mathematics; Houghton Mifflin (1972), Modern School Mathematics; Ginn (1972), Elementary Mathematics; and Silver Burdett (1973), Mathematics.

Finally, two recent texts were reviewed. They are: Scott, Foresman (1975), Mathematics Around Us, and Science Research Associates (SRA) (1974), Mathematics Learning System.

These texts were chosen as being representative of the many available texts on the market. Inclusion or exclusion of a particular series does not indicate a value judgement of the texts.

PURPOSE OF THE REVIEW OF THE TEXTBOOKS

The intent of the reviewer was to compare the treatment of initial fraction concepts by the various authors to the hierarchy which had been devised for this study. There was no attempt to place a value judgement on the presentation of fractions within any one text, or from text to text.

For purposes of simplification, the subconcepts of the hierarchy were roughly grouped into three categories. Category 1 consisted of prerequisites for learning fraction concepts which included concepts I through V. Category 2 consisted of initial fraction instruction--concepts VI through X. Category 3 dealt with combining, ordering, and equating fractional parts, and included concepts XI through XVI.

A summary of the general treatment of fractions within each text is given. Finally, the philosophy of teaching mathematics, as indicated in the preface of each textbook, is given.

This data is organized in a chart which comprises the main part of this chapter. An attempt at evaluating trends in teaching initial fraction concepts, as indicated by the several texts, follows the chart.

THE CHART

Chart 2 indicates how each textbook series treats the concepts within each of the categories of the hierarchy.

CHART 2

TEXTBOOKS

Category 1. Prerequisites to learning the initial fraction concepts.

Counting, same number as, <, >, =, length, area volume, same length as, same area as, same amount as, meaning of "unit," using units, measurement. Parts from wholes, subsets as distinguished from sets, equivalent partitionings, reconstruction of the whole after being given the parts.

SMSG: Most of these topics are covered in grade K and grade 1. Activity approach. Sets, >, <, =, premeasurement activities. Number line, regions. Partitioning of sets (10 pgs.), regions (2 pgs.). Grade 2--More work on measurement. Grade 3--More work on regions (20 pgs.).

Holt, Rinehart, Winston (1966): Grade 1--Most prerequisites covered. Recognition of congruent parts of the whole (region and sets); words "half," "fourth." Measurement (4 pgs.); time (6 pgs.), using "half past the hour." Grade 2--Measurement (10 pgs.). Equivalent partitioning (1 pg.).

Harcourt Brace (1972): K--"Half;" measurement (non-fractional), larger, smaller, number line. Grade 1--"Greater than," "less than;" measurement, linear, volume. Grade 2-->, <, =. Regions.

CHART 2 (CONTINUED)

TEXTBOOKS

Addison Wesley (1973): Grade 1--Greater than, less than, >, <, =; sets, strips, counters, number line, symmetry; inch trains, cm. trains; same area as. Grade 2--Regions; congruence, length, shape.

Houghton Mifflin (1972): K--Ordering, equal shapes, equivalent sets; measurement, largest, smallest, same size; one-half; regions (1 pg.), sets (1 pg.). Equivalent and non-equivalent partitioning. Grade 1-- >, <, =; number line, measurement, regions. symmetry. Grade 3--Area, volume.

Ginn (1972): Grade 1-->, <, =; sets, subsets, readiness for number line, measurement: linear, area, clock. Number line, ordered pairs, coordinates. Equivalent partitionings of halves (no symbols), region (2 pgs.). Grade 2--Recognize congruent division into halves, thirds, fourths, fifths, as distinguished from non-congruent partitionings. (1 pg. area; 1 pg. linear).

Silver Burdett (1973): K--Greater than, less than. Equivalent and non-equivalent partitionings, halves and thirds, number line and region (10 pgs.). Grade 1--<, >, =; measurement (1 pg.). Mark halves, thirds, fourths of drawings (region, 4 pgs.). Grade 2--Symmetry and congruence. Grade 3--Area and volume.

Scott Foresman (1974): K--Halves, thirds, fourths (region). Grade 1--Measurement (length, volume, time); <, >, =; meaning of unit measurement; equal parts, non-equal parts (region).

CHART 2 (CONTINUED)

TEXTBOOKS

SRA (1975): Grade 1--Measurement. Grade 2-->, <, =. Number line. A lot of geometry.

Category 2. Initial fraction instruction.

Presymbolic use of words, use of symbols (1/2), non-congruent fractional parts, counting equivalent fractional parts, constructing fractional parts if given the whole, interpreting the whole if given fractional parts.

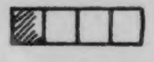
SMSG: Grade 1--Initial fraction instruction, including "color 1/3 of the set." (8 pgs.). Grade 2--More of the above, plus fractions on the number line. Grade 3--Repeat initial fraction instruction on regions, sets, number line (20 pgs.). Constructing fractional parts (1 pg.). Grade 4--More work on above (10 pgs.).

Holt, Rinehart, Winston (1966): Grade 2--Unit fractions (5 pgs.); counting fractional parts (2 pgs.). Number line, time. Grade 3--Initial fractions (20 pgs.); region, number line; sets. Different ways of partitioning a region into congruent parts. Measurement with fractions on ruler (5 pgs.).

Harcourt Brace (1972): Grade 1--Initial fractions (4 pgs.); region, set. Time. Grade 2--1/2, 1/4, etc. (5 pgs.), region, set. Clock (use of 1/2). Grade 3--Initial fractions (sets, region, number line).

CHART 2 (CONTINUED)

TEXTBOOKS

Harcourt Brace (Cont.): Use of ordered pairs:  (1,4); then after ordered pairs, interpret as 1/4.
Applications: measurement, time, ruler, temperature.

Addison Wesley (1973): Grade 1--"Paste in the missing half," third, fourth. Region (8 pgs.). Grade 2--Paste in 3 of 4 parts; symbols (3/4); activities (strips, sets, regions) (9 pgs.). Grade 3--Linear measure (3 pgs.) (2 1/2 in.); region fractions (part-whole) (4 pgs.). Grade 5--Non-congruent parts.

Houghton Mifflin (1972): Grade 1--"Color one-half" regions; equivalent and non-equivalent partitionings (8 pgs.). Sets (2 pgs.). Use of 1/2, 1/4. Grade 2--3/4, 2/3 sets, number line (5 pgs.). "Measure to nearest 1/2 in." (3 pgs.). Time (2 pgs.). Regions (9 pgs.). Grade 3--Initial fraction concepts (whole chapter). Use of sets, regions, number line. Grade 4--Sets, region, number line (8 pgs.). Interpreting whole, given a fractional part $b(\frac{1}{b}) = 1$. (1 pg.).

Ginn (1972): Grade 2--Introduction to 1/2, 1/4, 1/5, etc. on number line. 3/5, 8/4, 5/5, etc. (4 pgs.). Grade 3--Region (1 pg.), number line (3 pgs.), coordinate place (13 pgs.); angles of a circle, divide it into 1/4, 1/6, 1/3, etc. (14 pgs.). Equivalent fractions from a "gum ball machine." Order of fractions on number line. Grade 4--Recognition of name of fractional part: apple, volume, sets, circles, dials, test tubes, measurement, probability; partitioning region or number line into 3/4. Grade 5--Interpret whole, if given fractional part.

CHART 2 (CONTINUED)

TEXTBOOKS

Silver Burdett (1973): Grade 2--Fractional notation: region (4 pgs.), ruler "nearest 1/2 inch" (2 pgs.). Grade 3--Initial fraction concepts related to region (7 pgs.), sets (1 pg.), and number line (2 pgs.).

Scott Foresman (1974): Grade 1--1/2, 1/3, 1/4; region (4 pgs.). Grade 2--1/2, 2/3, 3/4; region (4 pgs.); sets (4 pgs.). Grade 3--Meaning of fraction; region, sets, number line (13 pgs.). Use of fraction in linear measure (1 pg.). Grade 4--Congruent and non-congruent fractional parts. Region (5 pgs.); sets, (2 pgs.); number line (2 pgs.), measurement (6 pgs.), mixed numbers (6 pgs.), including region, set and number line. Idea of "= to 1."

SRA (1975): Grade 1--1/2, 1/4, region (7 pgs.), including: 1/8, 2/4, 4/8. Grade 2--1/2 (volume), region mostly (9 pgs.). Symmetry (1 pg.). Partitioning regions (1 pg.). Grade 3--Initial fraction instruction: region (4 pgs.), sets (2 pgs.), number line (9 pgs.). Interpret whole, given fractional part (1 pg.).

Category 3. Combining, Ordering, and Equating Fractional Parts

Combining and separating fractional parts, equivalent fractions, ordering fractions and comparing, and combining fractional parts with unlike denominators.

SMSG: Grade 3--Ordering fractions by use of region and number line (4 pgs.). Grade 4--Equivalent fractions (16 pgs.). Grade 5--Combining and separating fractions with regions and number line (20 pgs.).

CHART 2 (CONTINUED)

TEXTBOOKS

SMSG (Cont.): Also, combining fractional parts with unlike denominators (5 pgs.).

Holt, Rinehart, Winston (1966): Grade 2--"Which is greater" (1 pg.). Grade 3--Equivalent fractions and ordering fractions. Grade 4--Equivalent fractions (6 pgs.) and ordering fractions (5 pgs.). Grade 5--Combining fractions using number line and region (1 pg.). Review of equivalent fractions (3 pgs.). Review of ordering fractions (1 pg.).

Harcourt Brace (1972): Grade 3--Combining fractional parts and separating; region; (3 pgs.). Equivalent fractions (2 pgs.); with sets (4 pgs.). Ordering fractions, number line, and regions (2 pgs.). Grade 4--Use of region, sets, number line for combining and separating, for equivalent fractions, and for ordering fractions (6 pgs.). Grade 5--Review above (6 pgs.). Combining fractional parts with unlike denominators (4 pgs.).

Addison Wesley (1973): Grade 4--Combining fractions on number line. Equivalent fractions on number line and using colored strips. Ordering fractions using region, colored strips, and number line. Combining fractional parts with different denominators, using colored strips. Grade 5--Combining fractions using number strips, regions, and number line. Equivalent fractions using colored strips, sets, and paper folding. Combining fractions with unlike denominators, using number strips, number line, and regions.

CHART 2 (CONTINUED)

TEXTBOOKS

Houghton Mifflin (1972): Grade 3--Combining fractions on number line, with bars, measurement, word problems, probability on a colored spinner. Ordering fractions using bars and number line. Equivalent fractions sets, region, bars, number line. Grade 4--Combining fractions using sets, regions, volume, number line, probability, word problems. Unlike denominators, combining fractions using paper folding and number line; extension to probability using sets of colored marbles. Equivalent fractions, sets, regions. This topic covered twice, in two different chapters. Grade 5--Combining fractions, sets, regions, number line, word problems. Equivalent fraction sets, regions, number line.

Ginn (1972): Grade 3: Equivalent fractions on number line; order of fractions on number line. Grade 4--Combining and separating fractions on number line (6 pgs.); ordering fractions on number line (2 pgs.). Grade 5--Combining and separating regions (2/7 + 3/7); equivalent fractions on number line; ordering fractions on number line.

Silver Burdett (1973): Grade 3--Equivalent fractions; regions and number line; fractions greater than one. Ordering of fractions by region.

Scott Foresman (1975): Grade 3--Comparing fractions; region. Grade 4--Equivalent fractions, paper folding, number line, sets (7 pgs.). Grade 5--Equivalent fractions, number line, region, measurement, area, applications to probability.

CHART 2 (CONTINUED)

TEXTBOOKS

SRA (1974): Grade 3--Combining fractions using regions and number line. Equivalent fractions using regions and number line. Grade 4--Review of Grade 3 plus ordering fractions using regions, sets, number line. Ordering fractions: comparing parts, when the wholes are different sizes; use of regions, number line; extension of this comparison of fractions to probability. Grade 5--Equivalent fractions; ordering; using regions, sets, number line and number strips.

Summary of Fraction Sequence in Text.

SMSG: Generally, SMSG devotes a large amount of space to the development of initial fraction concepts. The representation of fractions seems to be mainly as region (part/whole) and number line (sub-unit/unit). Measurement is used as an extension of fraction work--fractions are included in linear, area, and volume measurement.

Holt, Rinehart, Winston (1966): Generally in Grades 1 and 2 the fraction concepts are few (mainly "readiness"). Grade 3 devotes a full chapter (20 pages) to initial fraction concepts (more written language). Grade 4 reviews this work and extends it to computations with rational numbers. Grade 4 is very business-like and uses fewer drawings of regions and number lines. Grade 5 reviews initial fraction introduction in 3 pages, using region and number line. Except for Grade 3, there is very little use of "concrete" pictures; mostly "computation."

CHART 2 (CONTINUED)

TEXTBOOKS

Harcourt Brace (1972): Multiple embodiment presentation using region, sets, and number line. Grade 2 has a lot of use of sets and number line for <, >, =, +, -, X, ÷. Grade 3 uses region and number line for ordering fractions. Grade 4 introduces +, -, using combining regions. Grade 5 uses number line and regions for +, -. Use of drawings all the way to Grade 6.

Addison Wesley (1973): Generally an activity approach is used in Grade 1, especially the use of centimeter strips. Fractions are introduced as regions; concurrently a great deal of work with linear measurement (non-fractional). Grade 2 also uses an activity approach. In Grade 3 there is a measurement approach to fractions in the first chapter (rulers, area). It seems a "surface" approach to fractions, almost as if the children should understand the number line without having seen fractions on the number line. In Grade 4, fractions are related to colored strips and number lines (Ch. 14). Grade 5 has three chapters on fractions, mainly measurement. Grade 6 has 3 chapters on fractions with multiple embodiment and applications: sets, regions, measurement, time, paper folding, word problems, number line.

Houghton Mifflin (1972): K--Informal development of fraction concepts; children talk about ideas. Grade 1--Use of laboratory materials; sets; number line. Grade 2--Fractions (25 pgs.). Grade 3--Multiple embodiment for initial fraction concepts; extensive use of drawings; minimal use of written matter. Generally an extensive treatment of initial fraction concepts with many extensions of a variety of applications.

CHART 2 (CONTINUED)

TEXTBOOKS

Ginn (1972): Ginn is introducing very advanced topics early in the program. Little is done to promote understanding of what a fraction is. It seems mainly to be treated as part of the rational number system very early in the program. An activity approach is used. Fractions and decimals and percents are used from the fourth grade on.

Silver Burdett (1973): The treatment of arithmetic in Grade K, 1 and 2 is mostly sets and number lines; bundles of sticks for regrouping. The book has the flavor of those written perhaps a decade earlier. Limited introduction of fractions in Grade 1 and 2. Grade 3 introduces fractions.

Scott Foresman (1975): Fractions are introduced by use of part/whole (region) embodiment, then sets, then the number line. Applications are generally to measurement. Grade 1 (4 pgs.); Grade 2 (8 pgs.); Grade 3 (13 pgs.); Grade 4 (30 pgs.). Scott Foresman uses decimal fractions and the metric system a great deal. Decimal fractions are used from Grade 4 on.

SRA (1974): This text teaches more of the details of the hierarchy (VIII, IX, X, XI, XII, XV) than most others. SRA has stated that one goal of the program is to develop the idea before using it. Generally, there is a review chapter which uses multiple embodiments of the initial fraction concepts. Later in the text a chapter is devoted to abstraction of the ideas to the rational number system..

CHART 2 (CONTINUED)

TEXTBOOKS

The philosophy of the authors of the series, as written in the preface of the text.

Some authors were more profuse than others, and the treatment here is a function of the treatment in the preface of each text.

SMSG: The authors stress understanding of basic concepts of the structure of mathematics and provide a firm foundation for the understanding and use of mathematics in a scientific society. They stress introducing concepts in concrete terms and then abstracting mathematical ideas. Further, they work from specific ideas to more general concepts.

Holt, Rinehart, Winston (1966): Pupils discover patterns and relationships between operations and number systems. Structural characteristics of these systems and mathematics generally are explored. The discovery approach is used. The pupil is an active participant in the learning process. Words, pictures, diagrams, activities and questions are important in learning mathematics. The pupil learns to understand and tell why something is so.

Harcourt Brace (1972):

1. All students can think, reason, and compute effectively in mathematics.
2. Good questions stimulate thinking.
3. New concepts are related to prior skills.
4. Simple language is used.
5. Strategies are taught for thinking through problems.

CHART 2 (CONTINUED)

TEXTBOOKS

Harcourt Brace (Cont.):

6. Practice is sufficient and varied.
7. Students will achieve pleasure and satisfaction:
    a. freedom to explore ideas.
    b. variety of materials and models
    c. work from concrete to symbolic
    d. teachers should accept pupils' efforts to learn as okay.

Students should receive positive reinforcement from the teachers.

Addison Wesley (1973): The Addison Wesley philosophy is based on:

1. Active participation from the child.
2. Exciting explorations and investigations.
3. The unfolding of the structure and beauty of mathematics.
4. Discovery method.
5. Looking for patterns and relationships.
6. Students will develop their own generalizations.
7. After developing an idea, there will be problem solving practice.
8. The teacher will provide for individual differences.
9. There are fundamental mathematical concepts which can be isolated and set forth with clarity and sharpness; these concepts, when fully understood, provide powerful tools for extending knowledge; children of every age level should be encouraged to actively participate, to think, to question, and to seek understanding; although a certain body of knowledge must be passed from generation to generation, the individual creativity of each new generation must not be stifled by pedagogy which is inadequate for future learning.

CHART 2 (CONTINUED)

TEXTBOOKS

Houghton Mifflin (1972): Kindergarten. Discovery is the approach used and is the key to involvement and understanding. A multi-sensory, varied approach is used. The child works from the concrete to the abstract. There is periodic reinforcement of skills and concepts. There is emphasis on keeping a record of observable behavior as a guide for further teaching and reteaching. Most kindergarten work is readiness for later work.

Upper grades.

1. Discovery: encourages students to be more perceptive, inventive, and self-reliant.
2. Process: students should not be expected to understand "completely" any topic the first time it is encountered.
3. Students learn best what they enjoy. Multiple approaches are encouraged: games, puzzles, experiencing success, easy vocabulary, and simple sentences.
4. Learning is an individual process.
5. Patterns help learning.
6. Concrete to abstraction of ideas should be repeated at different levels of maturity.
7. Drill reinforces concepts and skills.

Ginn (1972): This program is designed to promote children's mathematical thinking, not their memorization of abstract patterns or manipulative "tricks." Emphasis is placed upon the foundation of young children's thinking—their perceptions. Physical models and applications are always present to aid the child with his abstract work. Through this technique of providing concrete references, it is possible to include more challenging mathematics while at the same time providing for differences in individual abilities and rates of achievement. The activity approach is used.

CHART 2 (CONTINUED)

TEXTBOOKS

Silver Burdett (1973): The Silver Burdett authors use a laboratory strategy. First a new idea is presented in a real world situation; then the idea is translated into symbols; then there is mathematical computation; then the problem is translated back into the real world. This is called "tell, show, solve, answer." Mathematics is fun. Problem solving is stressed. The "why" and "how" of an algorithm is taught and reviewed.

Scott Foresman (1975): The general teaching strategy of the Scott Foresman series is based on four points:

1. motivate (use activities)
2. teach (do pre-book activities)
3. practice (use book)
4. apply (use games, races, and related activities)

SRA (1974): The SRA series focuses on the real world and develops many concepts from real world situations. The text spends more time developing an idea and its related skills before moving on than previous texts did. THINK questions are asked, which may have no answer, or may have many answers. Informal language is used. Three processes:

1. Exploratory--play with a big idea.
2. Instructional--acquiring skills.
3. Review and then explore applications.

Mathematics is relevant and vital; useful and interesting; relax and enjoy it. People should talk together about it. Mathematics does not have to be formal and abstract. Everyone should succeed in mathematics. Manipulate real objects. Involve parents at home and at school. An enthusiastic teacher helps.

EVALUATION OF TEXTBOOK TREATMENT OF INITIAL FRACTION CONCEPTS

Although it was not surprising to discover that the overall treatment of initial fraction concepts followed the hierarchy, it was reassuring to those who had constructed the hierarchy to see that their "logic" appeared to agree with the "logic" of so many mathematics educators. The surprising result was that the hierarchy actually extended over several years (usually kindergarten through fifth grade) in the instructional hierarchies which the authors of the texts had developed.

Some texts did not treat those fine details of the hierarchy which the writers of it had so carefully included, so as to make the hierarchy very complete. But our hierarchy was not intended to be an instructional one. It was mainly a logical hierarchy, written within the constraints previously detailed in Chapter 2 of this paper.

The texts generally treated the hierarchy sequentially. Category 1 was usually covered in kindergarten and grades one and two. Category 2 was covered in grade three most often, but occasionally overlapped with grades two, four, and five. Category 3 was most often treated in grades four and five. Some texts did not treat the initial fraction concepts of Category 3 from a concrete approach, but rather jumped to the use of rational numbers by then, and simply assumed that the children were able to transfer the knowledge of "combining" and "separating" which they had gained in grades one and two, when they had learned combining with whole numbers.

The most common initial fraction instruction was implemented in the part/whole embodiment. This was frequently followed by either subset/set or sub-unit/unit, or both. The most common extensions were to measurement and especially the clock (no doubt because learning to "tell time" is a common skill to be learned in the primary grades).

The most unique approach to the part/whole interpretation for initial fraction instruction seemed to be the Ginn approach. Although initial fraction instruction had been introduced in grade two on the number line, in grade three a unique part/whole approach was used. The authors had already introduced several angles (a 90°, a 45°, a 180°, and a 60° angle). They then proceeded to ask the children what part of a circle would be shown if a 90° were placed in it. Also, what fraction of a circle would be shown if a 90° plus a 45° were placed in the circle. This is partitioning of a sort we had not considered. Of course the angles were colored, and referred to in the text as "one yellow angle and one red angle."

Some texts tended to treat fractions as just a stepping stone toward the use of decimal fractions, but this was rare. Some texts, not limited by the restrictions on our hierarchy, moved more quickly to abstraction of the fraction concept, and integration of that concept into the idea of the rational number system. However, it was gratifying to note that most of the texts did include some return to the concrete derivation of the fraction concept with which they were dealing at the time, even though the text was for the fifth or sixth grade.

V

REPORTS OF FIVE PILOT STUDIES

The following chapter consists of reports of pilot studies submitted by graduate students in partial fulfillment of course requirements of the seminar on research in mathematics education offered by the Department of Curriculum and Instruction. Part of the purpose of the seminar was to lay the ground for a status study of children's understanding of fractional parts of wholes. After reviewing research and textbooks on initial fraction concepts, the graduate students in the seminar split into four groups of two and prepared pilot studies to investigate various aspects of initial fraction understandings as well as to test the efficacy of different research techniques.

Blume and Mitchell were interested in the language children typically used, prior to formal instruction, to speak about part-whole partitioning, and initial fraction situations. They decided to study the possibilities of using a game to get children involved and talking in these situations. Burke and Schuerger were interested in developing tasks that would tap a child's understanding and use of the criteria for determining whether something had been divided into certain types of fractional parts. Hiebert and Tonneson were interested in children's part-whole understandings, relative to fractions, as exhibited by their partitioning behaviors. Using Piagetian type research procedures, they sought to determine whether partitioning activities could reveal, in the set/subset case, a similar structure of part-whole understandings as that exhibited in the continuous quantity case. Kraus and Konsin, interested in children's initial understandings of part-whole comparison situations, investigated the feasibility of measuring these understandings without depending on symbolic fractional notation but rather on diagrams and regions.

The pilot studies were conducted in May 1977. Subjects were drawn (not at random but by volunteering) from various elementary schools around Madison. All proceedings were videotaped. An abstract of each study will now be presented followed by the studies themselves. A summary of some of the conclusions will follow at the end of the chapter.

PILOT STUDY 1

Blume and Mitchell studied the efficacy of using a game to determine a child's vocabulary specific to some area of mathematics as well as to measure in some degree the child's understanding of certain mathematical concepts. The authors felt that the game situation would be conducive to getting children to present their thoughts in a vocabulary that they would normally use with their peers concerning certain problems. The game Partopoly (from DMP) was used because the game tasks were closely related to the development of an initial concept of fractions in the set/subset context. The game was played by three different groups of second graders (three children in each group). Group playing times varied—25 minutes, 30 minutes, and 45 minutes.

In general, there was a disappointing lack of verbalization. There was very little interaction between either the children or the children and the adults. No unique language regarding fractions was exhibited. Three of the nine children showed a basic understanding of 1/3 and 1/4 of a set while the rest either did not know what to do or else fell into common errors (unequal sized parts). It was also noticed that children very often indicated a trend toward considering three objects as a third and four objects as a fourth, a result similar to that found by Burke and Schuerger.

The authors felt that the main reasons for the failure to obtain relevant verbalizations lay in the facts that the game may have forced a new vocabulary on the children and the game surroundings (the taping room, the sound of the cameras, unfamiliar adults, etc.) were unusual.

PILOT STUDY 2

Burke and Schuerger sought to examine what children in second grade understood by "halves," "thirds," and "fourths" in the context of sets and regions. Even though most second graders are exposed to some instruction in fractions, it was felt that information about initial understandings and misconceptions concerning fractions (relevant to the status question mentioned earlier) could be gained using these subjects. Of primary concern was determining what criteria (equal parts, number of parts, congruent parts) the children used in judging a given set or region to be divided into specified fractional parts. Eight children near the end of second grade were interviewed using a fixed format consisting of 19 yes or no questions and several impromptu questions. Children were asked to explain each answer.

Results showed that in general the children understood that a whole had to be broken into four parts in order to be divided into fourths, three parts for thirds, and two parts for halves. On the equal parts question, about 50 percent of the children seemed to understand the appropriate criterion and apply it correctly. For the congruent parts question, most children seemed to have no trouble with non-congruent fractional parts. (However, there was some reservation about the tasks used to determine this.) Regions and sets seemed to be of about the same difficulty. On the unit/sub-unit question, most children were not sure about the sub-unit when asked to circle one-fourth or shade two-thirds of a whole. There was also the common mistake that any subset of three was a third and any subset of four was a fourth. The results suggest that children may be able to decide when something is divided into thirds, fourths, etc. and yet not know how to "count" fractional parts, i.e., describe something as one-third or two-fourths. Finally the results seem to show a significant gap between the child's ability to recognize fractional partitionings and his or her ability to construct fractional partitionings.

The authors concluded that by the end of second grade children have already or are capable of progressing a long way towards identifying, naming, and constructing fractional parts of wholes under both region and set representations.

PILOT STUDY 3

Hiebert and Tonneson sought to examine equivalence partitioning in young children with respect to areas, lengths, and sets. The goal was to determine whether Piaget's description of the fundamental notions of fractions held under these three representations. Piaget identified seven fraction components or subconcepts which unite to form the mental structure underlying understanding of a fraction under the part-whole interpretation. From this structural unit emerges an anticipatory scheme which guides the child's actions throughout problem solving processes involving equivalence partitioning. Of the seven fraction components of part-whole understanding two were very primitive (all children examined could be expected to possess them) and a third component wasn't appropriate for the limited scope of this study. The components used in this study were

1. a fraction implies a determinate number of parts (a whole is divided, or conceived as divided, into a certain specified number of parts);
2. the subdivision must be exhaustive;
3. all parts must be equal;
4. the whole is conserved under subdivision.

The anticipatory scheme, which Piaget believes to be the crucial mechanism in the development of the fraction concept, implies that the child will anticipate what the end product of an equivalence partitioning is to be like with respect to part-whole and part-part relationships. This anticipatory scheme relies on the presence of the components of the part-whole "structural unit" just described. For example, when a child has to divide a cake into three equal pieces he or she must anticipate before making the first cut where he or she will make the second cut so that there will be three equal pieces with nothing left over.

The questions posed by the study were whether or not this anticipatory scheme holds for the equivalence partitioning of sets and lengths and whether a child's understanding of part-whole and part-part relationships involved in such partitionings can be described and experimentally determined in terms of the components of the structural unit mentioned before.

The task formats and protocols closely followed those employed by Piaget. The tasks required the child to divide up a quantity among a number of stuffed animals seated around a table so that each animal got the same amount, and so that the quantity was completely used up. The area task used a circular clay "pie," the length task a piece of licorice, and the set/subset task penny-candy (twice and four times as many candies as animals).

Nine children, ages five years four months to eight years, were each administered a task consisting of three parts (each part dealing with the same fractional number and using a different interpretation of fractions--set, line, or region). Partitionings involving halves, thirds, fourths, and fifths were investigated by varying the number of animals seated at the table. All tasks were individually administered in an interview situation.

The three physical interpretations of fraction produced tasks of various difficulties. The set/subset case was significantly easier than the area or length cases. Six of nine children were immediately successful in the set/subset task, while only two children succeeded in both the length and area tasks. Several factors may help to explain this result. Complete and immediate solutions to the continuous quantity tasks require well developed anticipatory schemes; the discrete quantity tasks do not require such schemes for solution. The later tasks were immediately solvable on the basis of number strategies, but the length and area tasks required subdivision into equal pieces.

A further result was that continuous quantity partitionings involving halves were easier than those involving fourths, which in turn were easier than those involving thirds suggesting that this is the sequence in which children master unit fractions. Under the discrete partitioning situations the predominant one-one partitioning strategy was used with equal success for all fractional divisions. Hence no sequence indicating an order of difficulty was observed.

The results of this pilot study suggest that the seven subconcepts identified by Piaget are appropriate only for describing children's part-whole understandings with respect to continuous quantity equivalence partitionings. However the strategies used by the children to solve set/subset tasks were so markedly different than the strategies employed in the continuous tasks that the seven subconcepts do not provide an effective framework through which one could describe children's part-whole understandings with respect to discrete quantity partitionings. Consequently a meaningful comparison of the part-whole understandings with respect to the area and length versus the set interpretations, as manifested through equivalence partitioning activities, was not possible.

Thus the authors see the central problem emerging from this study as being one of identifying criteria which define a complete part-whole fraction concept across all physical interpretations.

PILOT STUDY 4

Kraus investigated the feasibility of measuring the ability of young children (first graders) to compare and order fractional parts without the use of symbolic fraction notation or the use of fractional terminology. A set of four tasks based on an area interpretation of fractions was developed by systematic variation of the three basic elements of a fraction: the unit, the denominator, and the numerator. A conservation of area task was also included. Three first graders and one second grader were individually interviewed and examined (15 minutes).

The author found that when comparing the fractional parts of two areas (two pieces of construction paper) the children in general did not pay attention to whether or not the two original pieces of construction paper were the same size (the unit) or how many pieces they were divided into (the denominator). What they did attend to was the "look" of the pieces compared and the number of parts of each area being considered (numerator value).

The author questioned whether formal instruction in fractions would have a significant effect on a subject's ability to solve the tasks presented and suggested that a cross-sectional status study using grades K-5 might determine this.

PILOT STUDY 5

The purpose of Konsin's study was to investigate the effect that visual stimuli have on the ability of fifth grade students who have finished formal symbolic instruction in fractions, to order and compare fractional parts of the same or different objects. Six tasks were presented to four fifth grade boys in individual interviews. The figures compared were either the same or different in the size of the unit, the shape of the unit, the numerator, or the denominator.

The students were successful in most cases in determining the reaction represented by the shaded parts. The subjects had no difficulty in comparing fractions with the same denominator or when one fraction could be reduced to another. (That situation occured in five of the six tasks.)

When the same figure was divided into a different number of parts where the shaded parts represented equivalent fractions, the students had no difficulty in providing an explanation of why the areas were the same. A similar result held when the figures were not the same size but did have the same shape. Much difficulty was encountered, however, when comparing the fractional parts of two figures in which the figures were neither the same shape or size.

The author concludes that fifth grade boys seem to be able to correctly compare fractional parts of figures when the figures have the same shape and size or when the figures have the same shape but different sizes. This is true whether the figures are divided into a different number of parts or the same number of parts. Fifth grade boys have difficulty comparing fractional parts of figures when the figures have different shapes and are not the same size.

PILOT STUDY 1

A PRELIMINARY INVESTIGATION OF THE USE OF A GAME TO IDENTIFY CHILDREN'S USE OF LANGUAGE RELEVANT TO FRACTIONS AND CHILDREN'S KNOWLEDGE OF THE CONCEPT OF FRACTION

by

Charles Mitchell and Glen Blume

An area often overlooked in education is the general knowledge that children have acquired outside of the school environment. Children often encounter mathematics in their preschool years and at times develop their own special vocabulary. If a teacher is aware of a child's vocabulary, then this teacher is in a position to build upon it, and if need be, modify it. A purpose of this study was to determine whether children have acquired a special vocabulary relating to some initial fraction concepts.

With the need for awareness of a child's vocabulary established, we next turned our attention to the various methods for obtaining this information. Adult interviewers could question the child, but it is possible that a child uses one language to talk to adults and another when addressing peers. If so, which mathematical settings allow children to interact with children? It was our belief that some form of mathematical game might be one of the most promising situations. Not only are children generally interacting among themselves in many games, but after they are familiar with the game, they may require little adult intervention. We hoped to show that a mathematical game could be used to determine a child's vocabulary specific to some area of mathematics.

For purposes of this investigation, a "game" will be defined as an activity with a definite set of rules, where participants either compete amongst themselves or against game tasks. The children must perceive the activity to be a game.

In this preliminary investigation it was decided to use a game which was recommended for use in a current elementary school mathematics series. "Partopoly," from Developing Mathematical Processes (DMP) (Romberg, Harvey, & Moser, 1974, 1975, 1976) was chosen for the initial investigation because game tasks were closely related to the development of an initial concept of fraction in the set/subset context.

A further purpose of the study was to explore the use of a game as a vehicle for assessment of children's knowledge. Children's performance on the tasks was believed to be a potential indicator of their knowledge of and misconceptions about the various topics related to fractions. In particular, the game required the players to manipulate sets of differing cardinality. It was hoped that students' understanding of the set/subset characterization of the part-whole model of fractions could be assessed.

DESIGN

Subjects

In three game sessions a total of nine second grade children, three in each group, played Partopoly. Second grade children were chosen because it was believed that they would have been exposed to some fraction concepts in school, but that this instruction would have been minimal. It was believed that younger children would have difficulty reading and understanding the game tasks, and older children would have attained mastery on too many of the game tasks for it to be a potential means of identifying the extent of their knowledge.

Description of Game

Partopoly can be played by any number of children, but children get to move more frequently with fewer participants. Each child begins play with the same number of counters and the object of the game is to have the fewest number of counters at the end of the game. Most of the spaces on the board direct the child to perform a task after which there is a possible exchange of counters with the banker.

A child rolling a die can land on one of five distinct task situations. The first type involves children in dividing a set of chips into thirds, fourth, fifths, sixths, or twelfths. For this to be accomplished, the chips had to be equally divided into the necessary subsets. Figure 1 is a picture of the game board and Table 1 is the sequence of game tasks.

The second type of problem situation is for the child to determine if he or she has an even or odd number of chips in his or her set. The third type of problem requires a child to divide a set into three, five, six, or seven groups. In this case a remainder is allowed and children are usually instructed to "give away one group," after successfully performing the task.

A fourth possible situation asks a child to make groups of two or eight with chips or to group his or her set by five or ten chips. The child is then asked to give away the remainder. The final situation, termed miscellaneous, includes the "start" position, free positions

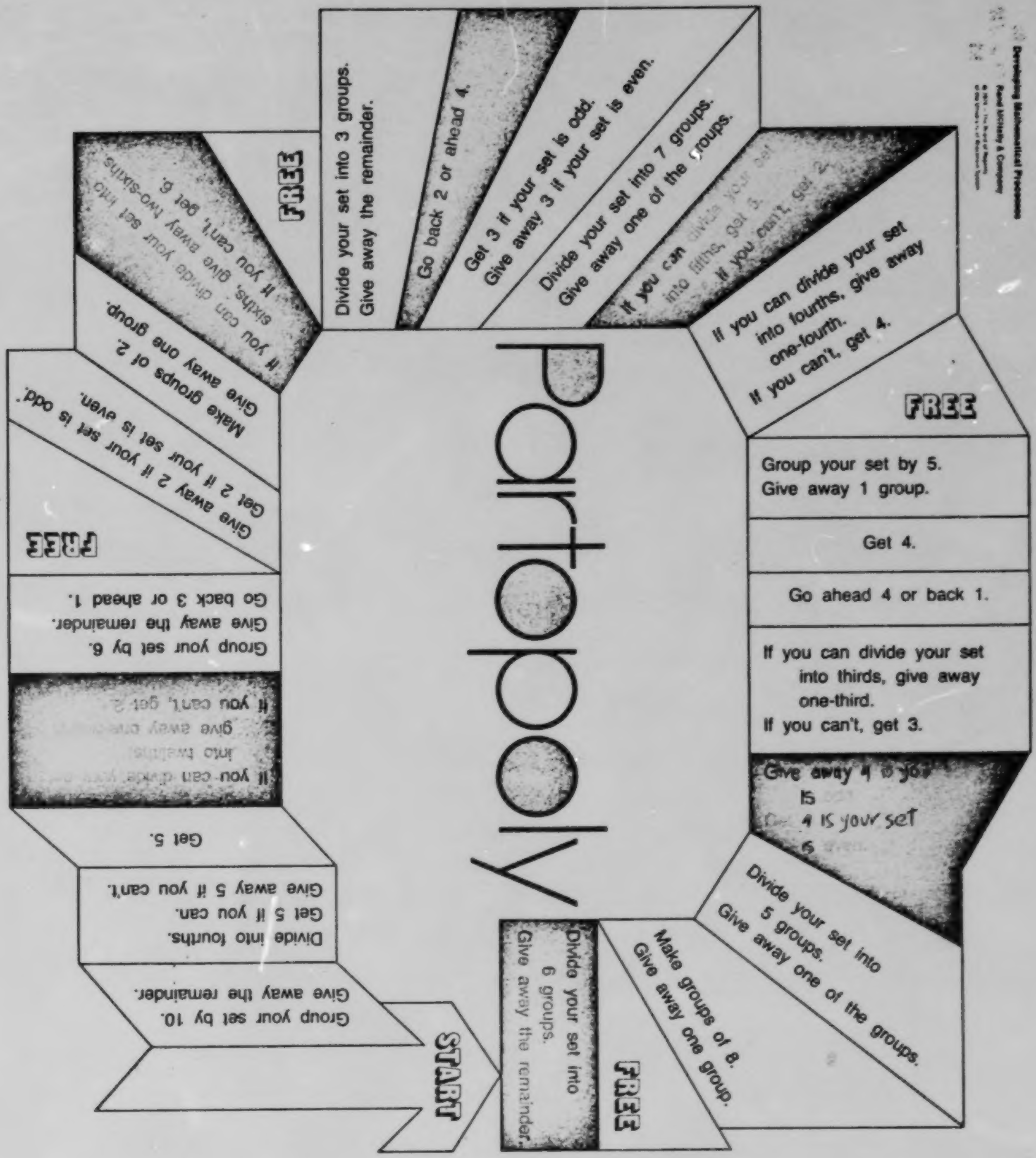

Partopoly
START
Group your set by 10. Give away the remainder.
Divide into fourths. Get 5 if you can. Give away 5 if you can't.
Get 5.
If you can divide your set into twelfths, give away one-twelfth. If you can't, get 2.
Group your set by 6. Give away the remainder. Go back 3 or ahead 1.
FREE
Give away 2 if your set is odd. Get 2 if your set is even.
Make groups of 2. Give away one group.
If you can divide your set into sixths, give away two-sixths. If you can't, get 6.
FREE
Divide your set into 3 groups. Give away the remainder.
Go back 2 or ahead 4.
Get 3 if your set is odd. Give away 3 if your set is even.
Divide your set into 7 groups. Give away one of the groups.
If you can divide your set into fifths, get 5. If you can't, get 2.
If you can divide your set into fourths, give away one-fourth. If you can't, get 4.
FREE
Group your set by 5. Give away 1 group.
Get 4.
Go ahead 4 or back 1.
If you can divide your set into thirds, give away one-third. If you can't, get 3.
Give away 4 if your
is
4 is your set
6
Divide your set into 5 groups. Give away one of the groups.
Make groups of 8. Give away one group.
FREE
Divide your set into 6 groups. Give away the remainder.
Developing Mathematical Processes
Rand McNally & Company

Figure 1. Partopoly gameboard.

TABLE 1

DESCRIPTION OF THE TASKS IN PARTOPOLY (IN ORDER OF OCCURRENCE)

Board Position	Task
0	Start
1	Divide your set into 6 groups. Give away the remainder.
2	Free
3	Make groups of 8. Give away one group.
4	Divide your set into 5 groups. Give away one of the groups.
5	Give away 4 if your set is odd. Get 4 if your set is even.
6	If you can, divide your set into thirds, give away one-third. If you can't, get 3.
7	Go ahead 4 or back 1.
8	Get 4.
9	Group your set by 5. Give away 1 group.
10	Free.
11	If you can, divide your set into fourths, give away one-fourth. If you can't, get 4.
12	If you can, divide your set into fifths, get 5. If you can't, get 2.
13	Divide your set into 7 groups. Give away one of the groups.
14	Get 3 if your set is odd. Give away 3 if your set is even.
15	Go back 2 or ahead 4.

TABLE 1 (CONTINUED)

DESCRIPTION OF THE TASKS IN PARTOPOLY (IN ORDER OF OCCURRENCE)

Board Position	Task
16	Divide your set into 3 groups. Give away the remainder.
17	Free
18	If you can, divide your set into sixths, give away two-sixths. If you can't, get 6.
19	Make groups of 2. Give away one group.
20	Give away 2 if your set is odd. Get 2 if your set is even.
21	Free
22	Group your set by 6. Give away the remainder. Go back 3 or ahead 1.
23	If you can, divide your set into twelfths, give away one-twelfth. If you can't, get 2.
24	Get 5.
25	If you can, divide your set into fourths. Get 5 if you can. Give away 5 if you can't.
26	Group your set by 10. Give away the remainder.

where no transactions occur, and board spaces where a child is automatically given four or five chips.

Two versions of the game were used. In all cases children were encouraged to help each other, but in one version a bonus rule applied during play. If a child could not successfully perform a task, or if an error was made, then other children could challenge the performance. If a child who challenged could verbally specify a correct procedure to be used for the task in question, he or she would be rewarded by being able to give the banker three chips from his or her set. Since the child with the fewest chips at the end of the game was declared the winner, it was believed that this version involving opportunity for a bonus would stimulate verbalization.

In the other version children were encouraged to freely verbally assist or correct each other after the individual had made his or her initial attempt at performance of the task. In each of the three sessions the children knew each other (were from the same school) and acted as though they were quite willing to help each other or, at least, to receive help.

During each of the three Partopoly sessions one of the experimenters explained the rules and played the "banker." It was the banker's function to settle disputes and, initially, to encourage interaction by reminding the children to either help each other or to invoke a challenge if a mistake was made. The other experimenter remained nearby to follow the children's progress around the game board and to begin surreptitiously coding responses. One author was "banker" for Groups I and II; the other was "banker" for Group II.

Group I played without the bonus rule for 30 minutes. Group II played with the bonus rule in force for about 25 minutes. Because there was insufficient verbalization, and because children in Groups I and II had time to make only two to four moves before time expired, the second half of the study was modified. The bonus rule was temporarily abandoned and the group played for 45 minutes. The number of chips with which a child began play was reduced from 40 to 20 in an effort to eliminate tedious counting and to allow the children to encounter more tasks.

Coding

Coding of the children's scores occurred in four steps. During the taping, one of the experimenters did preliminary coding of performance and verbalizations. Each of the experimenters then independently viewed the tapes and coded performance and verbalization. The experimenters disagreed on only five of the 51 tasks requiring coding, and the final step was the joint resolution of these five conflicts.

The coding included whether the performance was right or wrong, whether it was accomplished with or without help, strategies used, and relevant verbalizations. No attempt was made to classify strategies used by the children; they are reported below in the findings of the study.

FINDINGS

Table 2 is a summary of performance by tasks for all children. Little can be said regarding global performance by item. Although

TABLE 2

SUMMARY OF PERFORMANCE BY TASKS FOR ALL CHILDREN PLAYING PARTOPOLY

	Thirds, Fourths, etc.			Odd-Even		
	-	0	+	-	0	+
Frequency:	4	3	6	4	0	6

Make groups of n			Divide into n groups		
-	0	+	-	0	+
5	1	1	3	2	3

All tasks		
-	0	+
16	6	16

Remainder existed			Remainder did not exist		
-	0	+	-	0	+
9	5	8	3	1	2

Scoring key: + = Correct without assistance
0 = Correct after assistance was given
- = Incorrect

the "Make groups of n" task was performed incorrectly five out of seven times, for all tasks, approximately half were and half were not performed correctly. In addition, as indicated in Table 2, performance seemed unaffected by the existence or non-existence of a remainder.

Table 3 illustrates the sequence of tasks encountered during the play of the game. For each task, a description is given which includes the child performing the task, his or her number of chips at the start of the turn, whether or not a remainder was encountered (when applicable), and the child's score on that task.

SUMMARY OF PERFORMANCE

The following is a summary of performance and verbalization by each individual.

Child A

Child A performed none of her four tasks correctly without help. Dividing a set into thirds and making groups of two were done correctly with help, and dividing a set into fourths and an Odd-Even task were done incorrectly. The task involving fourths followed the task eventually completed correctly which involved thirds.

Child A formed 13 groups of three in response to the task of dividing her set into thirds, and after suggestions and questions from child C and the experimenter (E) to form three subsets, she persisted in retaining separation into groups of three within each subset during the process of consolidation into three sets. She somewhat

TABLE 3

SEQUENCE OF TASKS ENCOUNTERED IN THE PLAY OF THE GAME

Child	Tasks (in order of occurrence)	Starting No. of chips	Existence of Remainder	Scoring
Group I (Child A, B, C)				
A	Divide into thirds	40	yes	0
B	Odd-Even	40	---	+
C	Odd-Even	40	---	-
A	Divide into fourths	43	yes	-
B	Divide into thirds	44	yes	+
C	Divide into fourths	44	no	-
A	Odd-Even	32	---	-
B	Divide into fourths	47	yes	+
C	Odd-Even	48	---	+
A	Make groups of 2	35	yes	0
B	Odd-Even	48	---	+
C	Free	45	---	
Group II (Child D, E, F)				
D	Divide into 6 groups	40	yes	+
E	Divide into thirds	40	yes	+
F	Divide into 6 groups	40	yes	0

Scoring Key: + = Correct without assistance
0 = Correct after assistance was given
- = Incorrect

TABLE 3 (CONTINUED)

SEQUENCE OF TASKS ENCOUNTERED IN THE PLAY OF THE GAME

Child	Tasks (in order of occurrence)	Starting No. of chips	Existence of Remainder	Scoring
	Group II (Continued)			
D	Divide into 5 groups	35	no	0
E	Divide into fourths	40	yes	0
F	Make groups of 8	36	yes	-
	Group III (Child G, H, K)			
G	Divide into 6 groups	20	yes	+
H	Free	20	---	
K	Divide into 5 groups	20	no	-
G	Odd-Even	18	---	-
H	Get 4	20	---	
K	Odd-Even	18	---	-
G	Divide into fourths	14	yes	-
H	Free	24	---	
K	Group by 5	14	yes	-
G	Divide into 3 groups	11	yes	+
H	Divide into fourths	24	no	+
K	Make groups of 2	11	yes	-
G	Free	9	---	
H	Free	18	---	

TABLE 3 (CONTINUED)

SEQUENCE OF TASKS ENCOUNTERED IN THE PLAY OF THE GAME

Child	Tasks (in order of occurrence)	Starting No. of chips	Existence of Remainder	Scoring
	Group III (Continued)			
K	Get 5	9	---	
G	Make groups of 2	9	yes	-
H	Odd-Even	18	---	+
K	Divide into fourths	14	yes	+
G	Get 5	9	---	
H	Divide into twelfths	20	yes	-
K	Start	9	---	
G	Start	14	---	
H	Divide into fourths	22	yes	0
K	Odd-Even	9	---	-
G	Odd-Even	14	---	+
H	Free	17	---	
K	Divide into fourths	13	yes	+
G	Free	10	---	
H	Get 4	17	---	
K	Divide into 7 groups	17	yes	-
G	Make groups of 2	10	no	+
H	Divide into 7 groups	21	no	-
K	Make groups of 2	15	yes	-

arbitrarily used five groups of three, five groups of three, and three groups of three, with one chip remaining. After her decision to "shorten the piles," child A formed three subsets, each with three groups of three chips, and a remainder of 13. Upon realization that another group of three could be added to each of the three subsets, she stated, "Here, they each get another third." Although she had more than three chips remaining (she had seven chips, after failing to distribute three to one of the subsets), she stated, "Now, these can't be used because there's one too many." She was obviously thinking yet in terms of groups of three, and only after a suggestions from E did she distribute chips one at a time to each of the subsets.

Without any intervening tasks requiring division of a set into thirds, fourths, etc., child A distributed chips one at a time to four subsets in her next task. However, she did not correctly recognize that a remainder signified that it was not possible to divide the set into fourths, in spite of having done so (with help) previously.

On her third task child A incorrectly determined that 32 chips was an odd number of chips, using the rule, "If the first number is odd, then it's an odd number." This rule was elicited by a question from E.

On her fourth task child A was required to make groups of two. This was child A's first encounter with this type of task. She placed the chips, by groups of two, into two groups, and after announcing that she could not perform the task, she added the remaining chip to one of the piles. Child A's answer to E's question about the difference between making groups of two and making two groups was, "Groups of two is two groups, and two groups is two groups." Child B then showed, on her own chips, the correct way to make groups of two, and only then did child A complete the task.

Child A responded verbally to questions and suggestions, but was generally quiet. She offered no verbal assistance when others were performing their tasks.

Child B

Child B performed all four tasks correctly, but offered no help to the other children. The two tasks concerning odd and even numbers were done by counting, then stating whether the result was even or odd.

On the tasks involving thirds and fourths, child B persisted in forming the three or four subsets by successively adding sets of three or four chips to each, then stopping when it became evident that each subset would not receive the same number of chips. Fortunately, for her procedure, in each case the number of chips present and the specified fraction resulted in less than a full group of three or four chips remaining for distribution.

Child C

Child C performed two Odd-Even tasks correctly and failed to divide a set of 44 chips into fourths. She offered assistance to another child on one task by demonstrating groups of two on her chips.

One of the two Odd-Even tasks was an exact duplicate of another child's previous task, and on the other, child C's strategy was to count, then state "odd" or "even." Her strategy on the Fourths task was to form and then increment the four subsets using groups of four chips. After distributing 32 chips among the four subsets, the remaining 12 chips were distributed, four, four, two, and two to the subsets, upon which she announced that she couldn't do the task.

Child D

Child D successfully divided her set of 40 chips into six groups on her first turn, but was unable to divide her set into five groups on the second turn. To form six groups she attempted to make groups of six (but the sixth had only five chips in it). After counting five chips in the remainder, she moved a chip from the subset with five chips to the remainder, then counted five in the subset, and gave away five chips.

Again, with the Five Groups task, she formed subsets of five chips each (she momentarily began with one chip in each of two subsets, but quickly shuffled the chips and began again using five chips in each subset). When she had 10 remaining, both child E and child F disagreed with her procedure and suggested she use (two) more chips in each group.

Child D verbalized several times in questioning another child, but these were elicited originally in response to questions from the experimenter.

Child E

Child E successfully completed a task concerning thirds and, with help, divided a set into fourths. Her process for thirds was forming three subsets, incrementing by one chip in each until the chips were exhausted. When one group had more than the other two she was convinced that set of 40 chips could not be divided into thirds.

Again, for fourths, four subsets were formed, each being incremented by one chip. Much confusion arose among all three children regarding the possibility of the existence or non-existence of a remainder when the set had been divided into fourths. After completing the task with 10 chips in each of four subsets, child E remarked, "I should have known that--because 10 plus 10 plus 10 plus 10 is 40!"

Child E verbalized often in response to questions from E, but she volunteered assistance to another child in only one situation. In this case she explained the composition of groups of eight by saying, "Make groups of eight like eight...chips in each group!"

Child F

Child F was unable to divide her set into six groups without help, and she was unsuccessful in making groups of eight. To divide her set into six groups she formed four subsets, used the remainder as the fifth subset, then realized that another subset was necessary. After removing two chips from each subset, she formed six subsets of five and attempted to give away 10 chips. Child D, who had had six chips per subset on the identical task earlier, noticed that there were five in each subset, and after discussion, child F completed the task correctly.

On the Groups of Eight task, child F seemingly attempted to form eight subsets. First, nine subsets of three were formed, the remainder and two subsets were given away. Then, after questioning by child D and a suggestion to put more chips in each, confusion prevailed, and child F's final comment was, "But I can't make eight groups out of that."

Child F verbalized several times during the course of confusion and discussion about other children's tasks. Most of the verbalization occurred in response to questions from E.

Child G

Child G performed several tasks correctly and several incorrectly. However, two of the three tasks done incorrectly were correctly performed on a later turn. In forming six groups, child G began with six subsets of two and remarked, "I should have an 'even number' in each group," and then formed six subsets of three and gave away the remainder.

The task of dividing his set into fourths was done incorrectly by child G only in that he did not recognize the significance of the remainder after the four subsets were formed. When child K encountered fourths and had four subsets of three with a remainder of two, child G asked, "Do you have four on that (pile)?" When told he had three chips in each group, G responded, "Then you can't do it."

Evidently he associated subsets of size four with fourths. When child H encountered the Twelfths task and child G attempted to help, child G mentioned "two-twelfths" after child H had formed subsets of two chips. Later, in response to E's question about fourths, child G stated, "That means he has to have four equal groups." Child G demonstrated knowledge of the "equal-sized subsets" concept of fraction in the set/subset representation of the Part-Whole model, but he also seemed to think that fourths required subsets of four chips, and so forth.

When forming groups of two with nine chips, child G decided that it could not be done because he had one chip left. When child K incorrectly formed groups of three when asked to form groups of two, child G was puzzled and stated, "I don't know." This same task was later performed correctly when the set contained 10 chips. Again, later, child G suggested, "He was supposed to divide--like two chips, two chips--and give away one of them (subset)."

The first Odd-Even task encountered was performed incorrectly when the set contained 18 chips. These had been grouped by child G on a previous turn into three subsets of six chips. After counting the number of subsets, 18 was declared to be an odd number. Even after child K stated, "I thought 'all together'," child G persisted with, "Yes, three is odd." Later when 14 chips were present, child G counted the chips (and got 13) and stated, "I have an uneven number."

Child G seemed to learn during the game, as evidenced by correct performance and statements later in the game. He exhibited mixed performance and comment on the fraction tasks, however. Child G was perhaps the most verbal of the nine subjects, offering advice even without questions from E.

Child H

Child H encountered only five tasks requiring performance. He divided 24 chips into fourths correctly, and later attempted to divide 22 chips into fourths by incrementing each of four subsets by one chip each time, but became confused by the unequal subsets of six, six, five, and five. Assistance was necessary here.

The task with twelfths was attempted using subsets with two chips. The task to form seven groups was initially attempted with six subsets with two chips each, then three chips each, and again the remainder was confusing. Neither of these tasks seemed to be understood very well.

On the Odd-Even task child H had three subsets of six chips and determined that this represented an even number. However, the process used was to count the number of chips in one of the subsets, and from the six it was determined to be even.

Child H was very quiet and offered no verbalizations. He also seemed easily swayed by the opinions of the other children.

Child K

Child K correctly performed two tasks in which remainders were encountered in dividing a set into fourths. Other than these, child K performed a variety of seven other tasks incorrectly.

On both of the Odd-Even tasks child K focused on one or more subsets. In one case he stated, "Mine is odd because it's (one subset is) three," and later, "One group is even...I have two sets of three and one set of four."

In dividing his set into five groups or seven groups, child K allowed large remainders to exist, and he seemed confused by this. For the tasks involving groups of two, one time he formed three groups of three, and the other time he allowed too large a remainder to exist. Similarly, for groups of five, four groups of three were formed.

Child K seemed aware of the concepts necessary for the tasks, but was perhaps confused by the grouping terminology. He twice offered correct advice to other children on Odd-Even problems and the Fourths task, but he offered incorrect advice ("just have groups of two...and if you have some left over it doesn't matter") on dividing 21 chips into seven groups.

DISCUSSION OF VERBALIZATION, PERFORMANCE, AND INDIVIDUAL STRATEGIES

In Group I there was little relevant verbalization among the children. At times there were exchanges between child and experimenter, but these were also minimal and were not enlightening as far as revealing special vocabulary. More interesting were some observations made of this group. One child successfully performed tasks involving thirds and fourths even after watching opponents fail twice at these tasks. No attempt was made by the child to help her opponent on either of the two previous situations nor was help offered later when the third child was unable to perform this task.

An especially interesting action occurred in Group I and was later repeated in Groups II and III as well. Each of the three children in Group I persisted in forming thirds or fourths by successively incrementing the three or four subsets by groups of three or four chips respectively.

Unfortunately for this group and Group II, each child had only a few opportunities to perform tasks. Each child in Group I performed at least one Odd-Even task, and the strategy employed was to count the chips and state the answer. One child stated a rule for determining the answer, "If 32 is the number of chips, then the set is odd, since three is the first number, and 3 is odd." This rule did not bother the other children who both correctly performed this task at later times.

None of the children had to divide their set into n groups and only one child was required to make groups of two. The initial strategy was to divide the set into two groups, but the child later corrected herself after questioning by the experimenter.

Unexpectedly, there was little verbalization between children in Group II, in spite of the bonus rule incentive to challenge and explain. The children either forgot or ignored the application of the bonus rule, although after encouragement by the experimenter, rewards were given on occasion. Children in this group were expected to be silent until the task was completed, and then to challenge, if appropriate.

Data were particularly lacking in this group. Only one child performed tasks concerning thirds and fourths, and this child was successful each time. Her process involved distributing one chip at a time to the correct number of subsets. There was one attempt at making groups of eight and, apparently, the child tried to make eight groups. Three attempts were made to divide a set into groups of five or six. It seemed that children thought there should be _n_ chips in each of the _n_ groups.

In Group III there was again a disappointing lack of verbalization. The major area of interest was once again the performance of the children on the game's tasks. One of the children was able to perform tasks involving thirds and fourths as long as no remainder existed and as long as there were at least two chips per subset. Another child did not realize each subset had to have an equal number after partitioning the chips.

For the Odd-Even task, all three children felt it only necessary to count one subset of the set to determine the answer. Only after intervention by the examiner, to settle a dispute, did one child count the entire set.

For the grouping tasks, one child did not know that a remainder was possible and stated that he could not perform the task if he got a remainder. There was no reason for the child to think this, but this problem did not bother the other two children. In this task, and when dividing counters into groups of _n_, two of the children did not realize that all of the chips must be used until there were an insufficient number to go around again.

IMPLICATIONS

For researchers interested in children verbalizing about initial fraction tasks to other children, the results of the study were somewhat disappointing. Little verbalization occurred without initiation from the experimenters, and no unique language regarding fraction concepts was demonstrated. The reasons for the lack of spontaneous verbalization are not clear. The children were friends in school, but the taping room was new and the whirrrr of the cameras was a constant reminder of the unusual surroundings. Perhaps a familiar school setting with a familiar "banker" would improve the situation. Another possible reason for the lack of verbalization is that, at least initially when reading the problem, the game forced a new vocabulary on the children. The children who were observed were not from a DMP environment, yet the vocabulary with its meaning in the game was specific to DMP to some extent.

In all cases it did not appear that the children were in a competitive situation. We doubt that these children felt that winning at Partopoly was very important to them. Possibly the use of other rewards for correct play or winning might alter the situation and lead to more relevant verbalization by the children. This would certainly be more feasible in a school setting as opposed to a laboratory setting.

When evaluating the suitability of using a game to assess children's performance in any given area of mathematics, it is necessary that all defining characteristics of a game be satisfied. In the opinion of the experimenters, the children did not perceive the situation to be a challenge either between themselves of between themselves and the game's tasks. Therefore, we hesitate to make any inferences regarding this purpose.

One interesting conjecture which seems reasonable in light of the findings is that many of the children's early understandings and misconceptions about fractions are shaped largely by their experiences with the area or region characterization of fraction. When a set/subset situation is encountered, children quite often seem to associate the size of the subset with the denominator of the unit fraction involved. For example, when dealing with thirds, if the cardinality of the set is three, no difficulty occurs because the situation is analogous to the final configuration in the area case. When the cardinality exceeds three, however, it may be the case that children attempt to incorporate groups of three within each of the subsets simply because it provides contact with three objects, which is the familiar configuration in the area situation.

Several children exhibited a disregard for the remainder when it was not possible for a set to be divided into thirds, fourths, etc. This is, again, due most likely to lack of contact with the set/subset characterization of fraction. Students whose primary contact with fractions has been in the area of context have simply not had enough experiences in which it was not possible to divide something into thirds; the regions with which they have worked have never been impossible to divide into n subregions. If one views it as important that children realize that the entire whole must be divided into parts under the part-whole conception of fraction, then it is important that children are given exposure to the set/subset characterization in addition to the region-subregion characterization.

Many suggestions are possible at this point. Some would include modifying or abandoning Partopoly or modifying or abandoning some types of problem tasks. Perhaps a different group of children should be studied to elicit verbalization. Although the results concerning verbalization were less than encouraging, some interesting observations regarding performance of tasks occurred, and it is our opinion that the topics of children's language and verbalization during the game experience may yet be a viable focus for future research.

PILOT STUDY 2

CHILDREN'S ABILITY TO RECOGNIZE CERTAIN FRACTIONAL PARTITIONINGS OF WHOLES USING SET AND REGION INTERPRETATIONS

by

Mary Anne Schuerger and Maurice Burke

PURPOSE

The question we hoped to consider in this study was: "What do children in the second grade understand by the terms halves, thirds, and fourths in the context of sets and regions?" This question was approached through the following sequence of questions:

1. (Division of the Whole Question) Do children recognize that a whole is made up of two halves, three thirds, and four fourths?
2. (Equal Parts Question) Do children recognize that the parts must be equal with respect to some attribute in order to say that the whole has been divided into halves, thirds, or fourths?
3. (Congruent Parts Question) Do the children realize that the parts do not have to look the same or otherwise be congruent in order to be called "halves," "thirds," or "fourths?"
4. (Set versus Regions Question) Do the children do better on fractional parts of sets than on fractional parts of regions? Are the relative difficulties a function of the type of questions asked and the type of fractions used?
5. (Unit-Sub-unit Question) Do the children realize what the unit is when they are asked to take a fractional part of something? Do they recognize what the sub-unit is when they are asked to take a fractional part of a whole?

RATIONALE

By the end of the second grade, children have usually had some formal exposure to fractional concepts. The amount varies from school to school, but generally, judging from textbooks widely used, the instruction does not go beyond identification, naming, and constructing of fractional parts of wholes. It is important for instruction to build on understandings that are already present in children. It is also important that the common misconceptions and difficulties that children have with respect to a certain subject matter be taken into consideration in teaching. Thus, although the children have had varying degrees of instruction in fractions, we hope that our study will shed some light on the understandings and misconceptions the children at the end of second grade have with regard to fractions.

PROCEDURE

The data was drawn from 20 minute interviews with eight second grade students. The protocol was the same for each subject and the questions were all asked in the same order. Each child was given a set of 19 tasks with "yes" or "no" answers. Explanations were required after each answer. After these fixed tasks were given, the examiner asked several impromptu questions based in part on the responses the child had made earlier. These impromptu questions were included because it was found (while piloting the 19 tasks with a second grader) that although one could do very well on the 19 tasks, fundamental misconceptions still could exist that the tasks really did not get at.

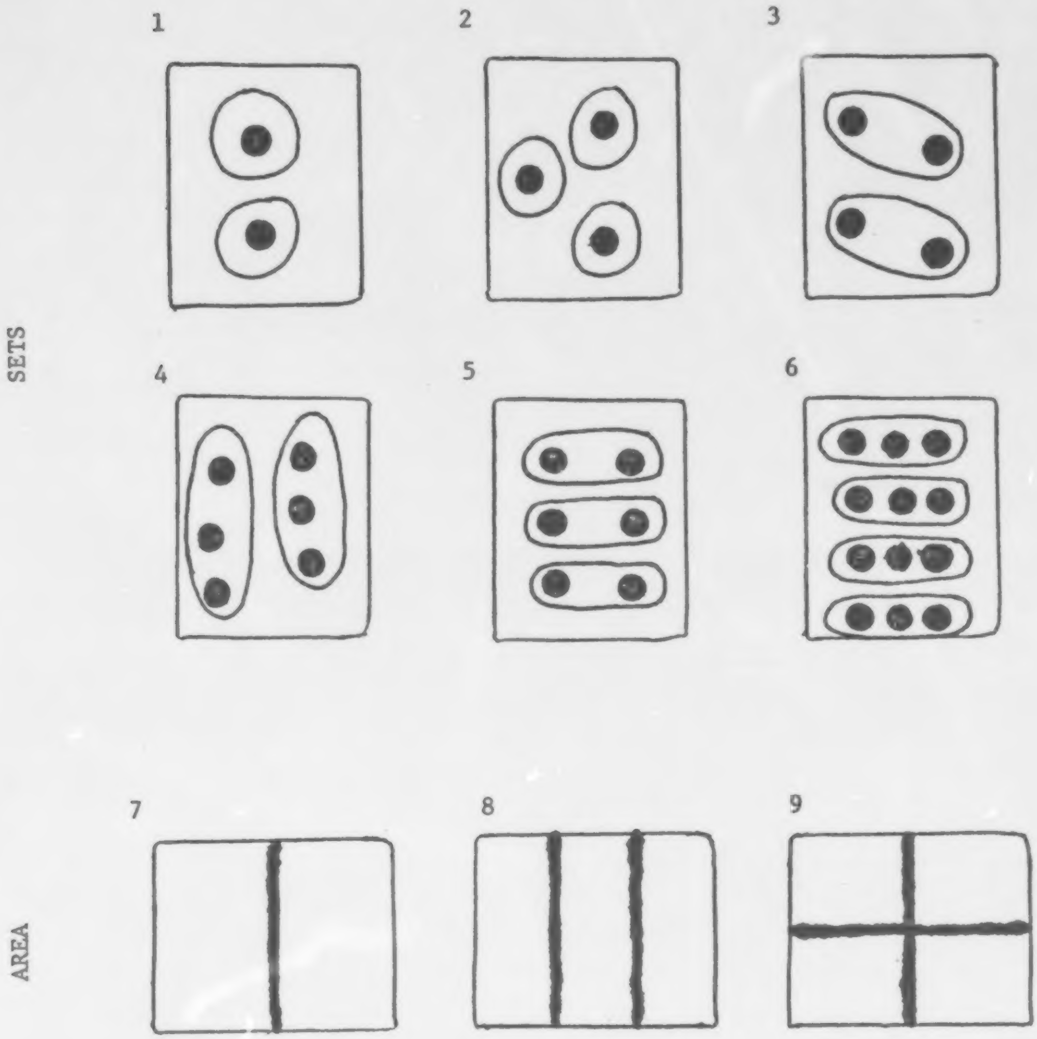
The 19 tasks were divided into three types, all involving sets and regions. In all tasks, the child was given a whole. The Examiner said: "Here is a picture of a set of _____/rectangle. Now I am going to divide this set of _____/rectangle up in a certain way, and this is how I do it." Using transparent overlays, the whole was then divided into parts. The examiner then asked: "Have I divided the set of _____/rectangle into halves/thirds/fourths?" The three question types centered on how the whole was divided. In one type, the whole was divided into congruent fraction parts. In another type, the whole was divided into non-congruent fractional parts. And in the final type, the whole was divided into non-fractional parts. (See Figure 2 for the task breakdown.)

Generally, for the impromptu questions, the child was given a set with six or eight elements or a rectangle divided into six or eight equal parts and then asked to circle or color one-fourth or one-third of the whole.

RESULTS

The selection of subjects was not random, but they did represent a variety of Madison schools. Fred came from Randall, Marc, Andy, and Bobby came from John Muir; Eric, Christine, and Kathleen came from Queen of Peace; and Scott came from Blessed Sacrament. Several children had been exposed to DMP materials on fractions. Others had used Scott-Foresman or Houghton-Mifflin. The math program of two of the children was not known.

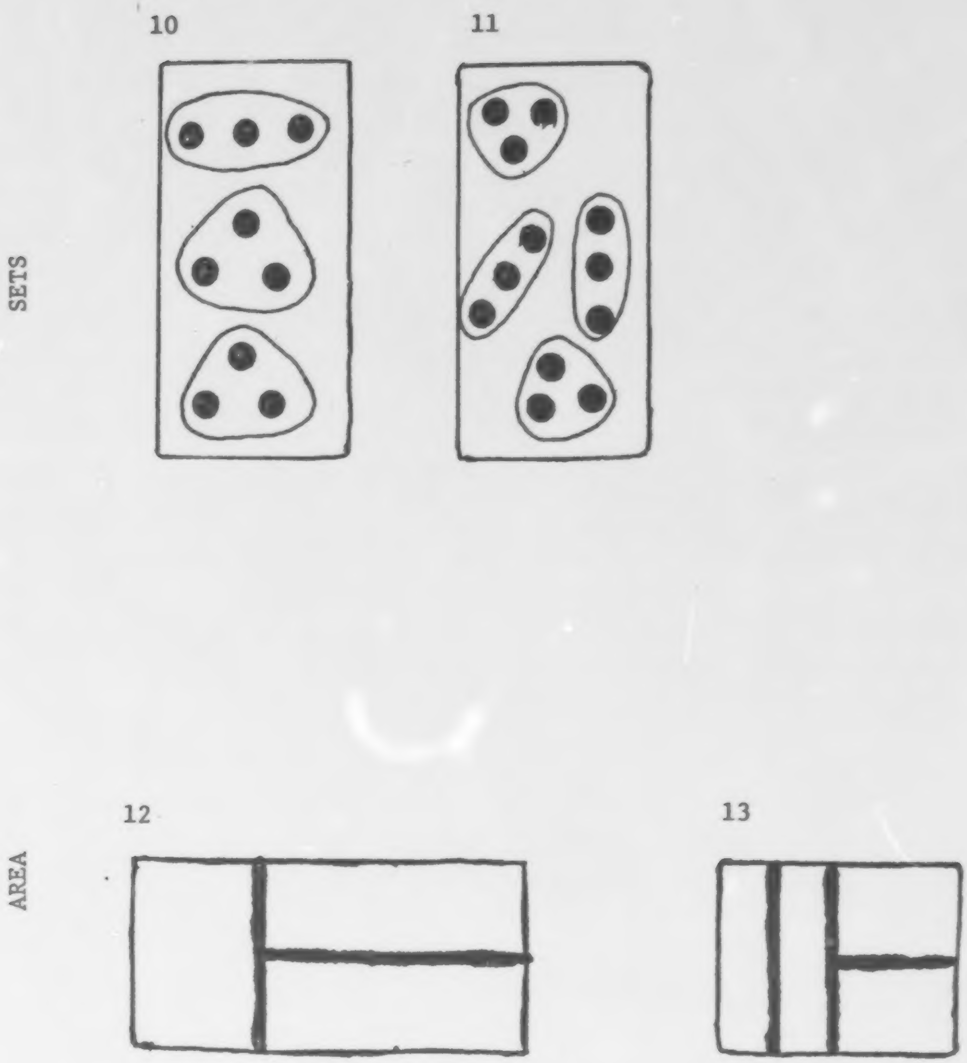
WHOLE DIVIDED INTO FRACTIONAL PARTS (CONGRUENT)



Note: The tasks in this chart were randomly ordered and then administered to the children. At first the children were shown the page with the dots, for example, and told, "Here is a set of 6 dots (using here Task 5 as an example). Now, I'm going to divide this set up in a certain way and this is how I do it." A transparent overlay was then placed over the dots, partitioning the set. The child was then asked,

Figure 2. Breakdown of the nineteen tasks: Tasks one to nine.

WHOLE DIVIDED INTO FRACTIONAL PARTS (NON-CONGRUENT)



"Have I divided the set into thirds?" A similar procedure was used for the area task.

Figure 2: Breakdown of the nineteen tasks: Tasks ten to thirteen.

WHOLE NOT DIVIDED INTO FRACTIONAL PARTS

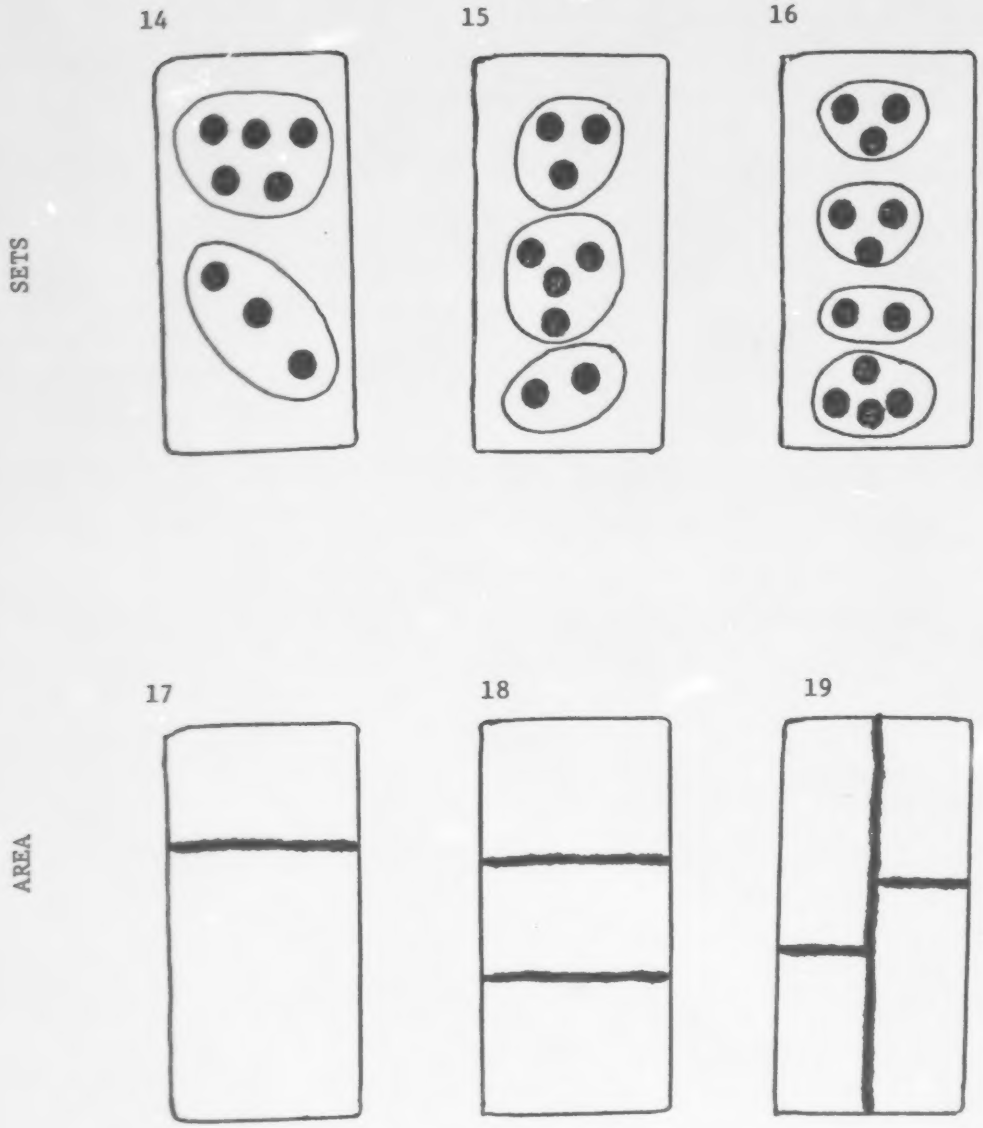

14
15
16
SETS
17
18
19
AREA

Figure 2: Breakdown of the nineteen tasks: Tasks fourteen to nineteen.

Because the responses are rather diverse, the results are listed individually. Fred seemed to know the criteria for identifying fractional parts. He answered every question correctly. His reasoning in Type 2 problems was that even though the parts differed in some dimension, they were the same size and hence were thirds, fourths, etc. He knew that a whole divided into fourths had to have four equal parts. (Likewise, thirds had to have three equal parts.) He was also able to determine one-fourth of eight dots and shade two-thirds of a rectangle divided into sixths.

Mark answered all the questions correctly except for one. He knew that a whole divided into fourths had four parts and a whole divided into thirds had three parts. He also knew that the parts had to be the same size (although they did not have to look the same). On the impromptu questions, he circled one-fourth of eight dots correctly and colored two-thirds of a rectangle divided into sixths correctly. His mistake came on judging a rectangle to be divided into thirds even though he knew the parts were not of equal size. This was not consistent with his correct application of the equal sizes criterion in the case of fourths.

Kathleen got all of the questions right except for the impromptu questions. She correctly applied the criteria of number of parts and size of parts in determining whether the wholes had been divided into halves, thirds, or fourths. On the impromptu questions, however, she failed to take into account the sub-unit. Given a rectangle broken into eighths, she colored one-eighth when asked to color one-fourth.

Given a set of six dots, she circled two when asked to color two-thirds, and one when asked to circle one-third. She seemed to be focusing on the number value in her constructions.

Bobby correctly answered all questions except for one of the designated tasks, but had some difficulty with the impromptu questions. He knew that a whole had to be divided into four equal parts to be divided into fourths and three equal parts to be divided into thirds. (Non-congruence caused no problems.) On sets, he vacillated in his criteria. He correctly applied the number of subsets/equality of subsets criterion on some tasks, but he confused his criteria in three different cases. In those cases, he felt that to be divided into thirds meant that each subset had three dots in it, and to be divided into fourths meant that each subset had four dots in it. In each of these three cases he eventually saw his mistake, but on the whole, he seemed confused as to which criterion to apply. On the impromptu questions, he circled three dots when given eight dots and asked to circle one-fourth. We did not know what to make of this. He colored two pieces when asked to color two-thirds of a rectangle divided into sixths. Also, he circled one dot when asked to circle one-third of a set of six dots. Thus he seemed to be focusing on the number value rather than the sub-unit.

Andy answered most of the questions correctly. He verbalized the correct criteria for fractional parts of sets and regions, and applied them correctly in most cases. However, on questions of Type 2, which dealt with area, he had some difficulty. Although he recognized the parts to be of equal size, he concluded that they were not fractional parts because they did not look the same. Some were longer than others and some were wider than others. He was not sure whether fractional parts had to look the same or not. On the impromptu questions, he was at first able to correctly circle one-third of six pennies, but when asked to circle two-thirds of six pennies, he became confused and circled all of the pennies. He reasoned that each third would have three dots in it (since a subset with three in it was a third). In his confusion, he seemed to revert to the criterion that a third was a subset of three. He had a similar problem with area. Asked to color one-fourth of a rectangle broken into eighths, he colored four of the parts. Again, rather confused, he concluded that one-fourth had four pieces and that two-fourths was the whole thing. He claimed that he was not able to color three-fourths of the rectangle.

Scott consistently applied the criterion that a whole had to be divided into three parts to be divided into thirds and four parts to be divided into fourths. Congruence or size did not matter. Even halves did not have to be equal. Hence he missed all the questions on non-fractional parts. In the impromptu questions, he was not able to circle one-fourth of eight dots but when asked to divide the set into fourths he separated it into four unequal subsets. On regions, he correctly colored one-fourth of a rectangle divided into eighths.

Christine consistently applied the same criterion as Scott. (A whole had to be divided into three parts to be divided into thirds.)

Size or congruency of parts did not matter. On halves, however, she seemed close to the equal parts criterion. She missed all of the questions of Type 3. On the impromptu questions, she concluded that one-fourth of eight dots was four, and one-fourth of a rectangle divided into eighths was four pieces. She decided that halves of a cake should be the same size and when presented with task 17 over again she managed to get it correct. She seemed to concentrate on the denominator in constructing her fractional parts.

Eric used the right criterion for judging halves of sets. There had to be two parts with equal numbers in both. For thirds and fourths of sets, however, he felt that a third was a subset of three and a fourth was a subset of four. With regions his criterion was the number of parts and not the size or congruency of the parts. He did voice some reservations when the parts did not look the same. In the impromptu questions he decided that halves of regions had to be the same size and correctly answered task 17 which was given to him again.

CONCLUSIONS

Although our sample was small, several things emerged from the results we obtained. First of all, for sets, most of the children's misconceptions polarized on two factors: (1) a set was divided into thirds if it was divided into subsets with three in each (likewise for fourths), and (2) a set was divided into thirds if it was divided into three subsets, regardless of size (likewise for halves and fourths). With regions, the main misconception was that a whole was divided into fourths if it had been broken into four parts, regardless of size or congruency (likewise for thirds and halves). Another common error was that if a region was cut into eighths and the child was asked to shade one-fourth of it, the child would color four parts since he or she felt that a fourth must have four parts. As far as constructions go, the main errors were the result of either focusing on the number value (one-third of six dots is one dot) or focusing on the denominator (one-fourth of a rectangle divided into eight pieces is four of the pieces).

The understanding of halves seemed to be better than the understanding of thirds and fourths. These latter two were about of equal difficulty.

In general, the children understood that a whole had to be broken into four parts in order to be divided into fourths, three parts for thirds, and two parts for halves. It should also be noted that over half of the children knew the equal parts criterion and applied it correctly in cases involving the recognition of fractional parts of given wholes. Congruency of parts did not seem to be a big factor. It was clearly a significant factor for only one child. One other showed signs of its influence.

All in all, we can answer the Division of Whole Question in the affirmative. There is a reservation, however. A child may think that a fourth is four equal parts of something whether that thing is a whole or is itself part of a whole. Thus seeing four equal parts, they judge the whole to be divided into fourths and, when asked to color one-fourth of eight pieces of a rectangle, they color four pieces of it. Our tasks were not sensitive to this possibility. For the Equal Parts Question, about 50 percent of the children seemed to understand the appropriate criterion and apply it correctly. For the Congruent Parts Question, most children seemed to have no trouble with non-congruent fractional parts. Regions and sets seemed to be about the same difficulty. It is hard to compare the two on this dimension since their relative difficulties seemed to be dependent on the nature of the problem. On the Unit-Sub-unit Question, most children were not sure about the sub-unit when asked to circle one-fourth or shade two-thirds of some whole. There was also the common mistake that any subset of three was a third and any subset of four was a fourth.

During the study it became evident that the tasks were not without problems. First of all, the congruence tasks could have been better. Although the parts did not look the same, they were nearly the same. Thus, a child might overlook the difference in giving his or her answer. Different pictures in some cases might have been useful. Secondly, the tasks did not seem to be sensitive to whether the child understood what one-fourth (one-third) of something was. At most they give some indication of what children understand "fourths" ("thirds") to be. The interview technique could have been greatly improved and follow-up questions could have been better.

A study of this sort could be of use to a teacher of second or third graders. First, very common errors that should be considered are of the form "a subset of three is a third" type of error, and "there are four parts in a fourth" type of error. Secondly, one must take note of the apparent gap that exists between the child's ability to recognize fractional parts and his ability to construct them. This gap probably reflects misconceptions and habits that go deeper than those distinguished in this study. Finally, as evidenced by Fred and Mark, children are able to progress a long way toward identifying, naming, and constructing fractional parts of wholes under both interpretations by the end of second grade. Hence, if time permits, much could be accomplished with second graders in the above three processes dealing with fractions.

PILOT STUDY 3

PILOT RESEARCH STUDY ON FRACTIONS

by

James Hiebert and Lowell Tonnessen

PURPOSE

This study examined the development of the fraction concept in young children. The fundamental notion of subdivision or partitioning of quantities was investigated in both continuous (area and length) and discrete (set/subset) cases.

RATIONALE

Piaget and his associates (Piaget, Inhelder, & Szeminska, 1960) have developed a rather comprehensive theory of the development of part-whole and fraction concepts. This theory is based on observations of young children placed in situations requiring the subdivision of a whole into a specified number of equal parts. Briefly, they suggest that between the ages of four and eight children acquire notions of part-whole relationships and the quantifiable notions of part-part relationships. These ideas are believed to be acquired gradually. Their development proceeds through several stages and is guided primarily by an anticipatory scheme which provides the child with the ability to recognize beforehand the part-whole and part-part relationships required in a subdivision problem.

Piaget's conceptual analysis of fraction, and his description of the development of the fraction concept seems intuitively reasonable and potentially beneficial for school instruction. However, several limitations of his work are apparent and these limitations suggest further investigations of the fraction concept. First, Piaget dealt with only one interpretation of the fraction concept--part of an area or region. It is conceivable that a different developmental pattern emerges within other interpretations--e.g., subset of a set. Second, all of the tasks employed in Piaget's work required actual subdivision by the children of a whole into a specified number of equal parts. The more common criterion in school arithmetic of a simple recognition of fractional part may produce significantly different results. Finally, Piaget claims that notions of fractional part-whole relationships and logical classification skills (as assessed by class inclusion and multiple classification tasks) develop in close synchrony, being logically and psychologically interdependent. However it is not clear that Piaget and associates administered both fraction and classification tasks to the same children, in which case the above claim remains a hypothesis.

The present study is restricted to a consideration of the first of these limitations--the physical interpretation of fraction. A logical task analysis by Novillis (1976) points out the distinction between the discrete and continuous quantity situations. And there is some empirical evidence (Campbell, 1975) which suggests that this distinction may be important in describing the developmental sequence of part-whole fraction concepts. However Piaget dealt only with the continuous case. An attempt was made here to partially replicate and extend Piaget's work by incorporating an interpretation of fraction in the discrete case (set/subset) and an additional interpretation of the continuous case (length).

RESEARCH DESIGN AND PROCEDURE

The sample for the present study consisted of six kindergarten children and six first grade children. Each child was individually interviewed for about 15 minutes and presented with several tasks.

For purposes of replication and comparison the task formats and protocols followed closely those employed by Piaget (Piaget et al., 1960). The general format for all tasks was to ask the child to divide up a quantity among a number of stuffed animals seated around a table so that each animal got the same amount, and so that the quantity was completely used up. The child's subdivision was followed by a conservation question about the quantitative equivalence of the parts and the original whole.

Three different tasks were used corresponding to three different physical interpretations of fraction. For the area or region interpretation the child was asked to divide a circular pie equally among the animals. A circle was selected since it seems to be the simplest regular shape which necessitates attention to area (rectangles, for example, can be considered from a linear perspective). For the length interpretation the child was asked to divide a string of licorice. And for the set/subset interpretation the child was asked to divide $2\underline{n}$ and $4\underline{n}$ pieces of candy equally among the $\underline{n}$ animals. The number $2\underline{n}$ was selected since it represented the simplest case requiring the logic of partitioning. However it is hypothesized that this task may be readily solved by intuitive, trial and error procedures. Therefore the task using 4n objects was also administered. It was hypothesized that the solution to this task would more likely require the adoption of a consistent strategy. The tasks investigated the understanding of the fractions half, third, fourth, and fifth by varying the number of animals from two through five.

Each subject received four tasks dealing with the same fractional number. These tasks covered the three interpretations of fraction as described above. Task assignments are shown in Table 4. (See Table 4).

This assignment of tasks permits a comparison of each of the three interpretations of fraction. Particular attention was given here to the various strategies employed by children in dealing with each interpretation. It also provides suggestive evidence concerning the developmental patterns of the fraction concept along the various interpretations. Piaget's work has demonstrated the developmental pattern for the area interpretation. These results should provide a comparison of this pattern with those using different interpretations.

Order of presentation effects were investigated by utilizing two different task orders. However due to the preliminary nature of this study and the small sample size, this consideration was given secondary importance. To minimize the confounding of order effects on changes across age and fractional numbers, two presentation orders were selected and assigned as follows:

TABLE 4

TASK ASSIGNMENTS

		Number of Subjects		
		Kindergarten	First Grade	Total
Fractional Number	Half	2	1	3
	Third	2	1	3
	Fourth	1	2	3
	Fifth	1	2	3

Define S_{ij} to be a particular subject where *S* represents grade level (kindergarten = K; first grade = 1); *i* represents a fractional number (i = 2, 3, 4, 5) and therefore specifies the set of tasks which that subject received; and *j* serves to index the subjects within a fractional number group (j = 1, 2 where 1 represents the first subject to be interviewed, and 2 represents the second subject to be interviewed). The two presentation orders were:

K_{21}---3142*
K_{22}---2413
1_{21}---3142
K_{31}---2413
K_{32}---3142
1_{31}---2413
K_{41}---3142
1_{41}---2413
1_{42}---3142
K_{51}---2413
1_{51}---3142
1_{52}---2413

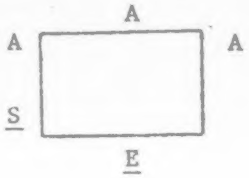
* Numbers correspond to task numbers as given in Task Protocols.

The rational for selecting these two orders derives from a conjecture that order effects would be most likely to occur between the continuous case tasks (e.g., the linear strategy of Task 2 may transfer to area--Task 1) and between the discrete case tasks (significant learning for Task 4 may occur in Task 3). Therefore, the order of these pairs of tasks were varied. Furthermore, to reduce the likelihood of a simple transfer of strategies from one task to the next, these task pairs do not occur in immediate succession.

Two classification schemes were used to code and categorize children's responses. The first scheme indicated the level of success on the fraction task and the conservation question. Subjects were classified as successful, transitional, or unsuccessful on each task. The second classification scheme employed the seven subconcepts identified by Piaget et al. (1960) as necessary for a complete understanding of fraction. These are: (1) the whole is divisible, potentially composed of separable elements; (2) a fraction implies a determinate number of parts; (3) the subdivision must be exhaustive; (4) there is a fixed relation between the number of parts and the number of divisions; (5) the parts have a nesting or hierarchical character; (6) the whole is conserved under subdivision; and (7) all parts must be equal. Procedures used to solve each task were analyzed in order to assess children's understanding of each of these subconcepts. This method of coding behavior places primary emphasis on the strategies used to solve the problems; the first classification scheme focuses more on the outcome or product of these strategies.

PROTOCOLS

Examiner (E) and subject (S) are seated with stuffed animals (A) around a small table as shown. E places a play-dough pie and a knife on the table.



Task 1 (Area)

If this is the first task, E asks S to determine how many animals are seated around the table.

LET'S PRETEND THESE ANIMALS ARE AT A PARTY AND THIS IS A PIE WHICH THEY ARE GOING TO EAT ALL UP. CAN YOU CUT THE PIE SO THAT EACH ANIMAL GETS EXACTLY THE SAME AMOUNT?

REMEMBER, EACH ANIMAL GETS THE SAME AMOUNT AND THEY ARE GOING TO EAT ALL OF IT.

E repeats instructions if necessary but does not alter them. After S has finished cutting, E says:

CAN YOU GIVE EACH ANIMAL THEIR PIE?

DOES EACH ANIMAL HAVE THE SAME AMOUNT? HOW CAN YOU TELL?

If it is impossible to tell on what basis S was dividing the pie, E brings out another and makes the first cut for S--then repeats instructions.

If S has some leftovers, E says:

REMEMBER, THEY ARE VERY HUNGRY AND WANT TO EAT ALL THE PIE. COULD YOU SHARE THOSE EXTRA PIECES SO THAT EACH ANIMAL GETS THE SAME AMOUNT?

E brings out another pie like the first one.

HERE IS ANOTHER PIE JUST LIKE THE ONE WE STARTED WITH. DO YOU THINK THERE IS MORE PIE HERE (WHOLE PIE) OR HERE (PIECES) OR IS IT THE SAME? WHY DO YOU THINK SO?

DO YOU THINK YOU'D GET MORE FULL IF YOU ATE ALL THE PIECES, OR IF YOU ATE THIS PIE, OR WOULD YOU BE JUST AS FULL EITHER WAY? WHY DO YOU THINK SO?

Task 2 (Length)

If this is the first task, E asks S to determine how many animals are seated around the table.

LET'S PRETEND THESE ANIMALS ARE AT A PARTY AND THIS IS A PIECE OF LICORICE WHICH THEY ARE GOING TO EAT ALL UP. CAN YOU CUT THE LICORICE SO THAT EACH ANIMAL GETS EXACTLY THE SAME AMOUNT?

REMEMBER, EACH ANIMAL GETS THE SAME AMOUNT AND THEY ARE GOING TO EAT ALL OF IT.

E repeats instructions if necessary but does not alter them. After S has finished cutting, E says:

CAN YOU GIVE EACH ANIMAL THEIR LICORICE?

DOES EACH ANIMAL HAVE THE SAME AMOUNT? HOW CAN YOU TELL?

If S has some leftovers, E says:

REMEMBER, THEY ARE VERY HUNGRY AND WANT TO EAT ALL THE LICORICE. COULD YOU SHARE THOSE EXTRA PIECES SO THAT EACH ANIMAL GETS THE SAME AMOUNT?

E brings out another piece of licorice.

HERE IS ANOTHER PIECE OF LICORICE JUST LIKE THE ONE WE STARTED WITH. DO YOU THINK THERE IS MORE LICORICE HERE (WHOLE LICORICE) OR HERE (PIECES), OR IS IT ABOUT THE SAME? WHY DO YOU THINK SO?

DO YOU THINK YOU'D GET MORE FULL IF YOU ATE ALL THESE PIECES, OR WOULD YOU BE JUST AS FULL EITHER WAY? WHY DO YOU THINK SO?

Task 3 (and 4) (Set/Subset)

If this is the first task, E asks S to determine how many animals are seated around the table.

LET'S PRETEND THESE ANIMALS ARE AT A PARTY AND THIS IS SOME CANDY WHICH THEY ARE GOING TO EAT ALL UP. CAN YOU SHARE THE CANDY SO THAT EACH ANIMAL GETS EXACTLY THE SAME AMOUNT?

REMEMBER, EACH ANIMAL GETS THE SAME AMOUNT AND THEY ARE GOING TO EAT ALL OF IT.

After S has given each animal their share E says:

DOES EACH ANIMAL HAVE THE SAME AMOUNT? HOW CAN YOU TELL?

E brings out a pile of candy like the first.

HERE IS ANOTHER PILE OF CANDY JUST LIKE THE ONE WE STARTED WITH. DO YOU THINK THERE IS MORE CANDY HERE (BIG PILE), OR HERE (PIECES) OR IS IT ABOUT THE SAME? WHY DO YOU THINK SO?

DO YOU THINK YOU'D GET MORE FULL IF YOU ATE ALL THESE PIECES, OR IF YOU ATE THIS PILE, OR WOULD YOU BE JUST AS FULL EITHER WAY? WHY DO YOU THINK SO?

RESULTS

Although the proposed study called for twelve subjects to be interviewed, only nine subjects were used, due to extra interview time required by several subjects.

In each of the tasks, the subject was judged to be successful, transitional, or unsuccessful. With respect to the tasks concerning fractional division of continuous objects (length and area), a subject was judged to be successful if he or she satisfied these four criteria: (1) divided the object into exactly n pieces (where n represents the number of animals), (2) did so on the first try, (3) cut pieces of approximately equal size, and (4) exhausted the whole.

A subject was judged to be transitional if he or she failed with respect to one of the four criteria, or seemed to understand the problem but did not know a strategy which would solve the problem.

A subject was judged to be unsuccessful if he or she failed with respect to more than one of the four criteria.

For tasks concerning fractional division of discrete objects (set/subset), a subject was judged to be successful if he or she exhaustively divided the whole into equal-sized portions in a systematic manner. A subject was judged to be transitional if the subject was eventually able to perform the task, but was either not systematic in dividing equally or did not exhaust the collection of objects without coaching. No subjects were judged to be unsuccessful on these tasks.

For conservation tasks, a subject was judged to be successful if he or she correctly answered the conservation question. A subject was judged to be unsuccessful if he or she incorrectly answered the conservation question.

The results of the categorizations based on the interviews are given in Table 5. Each subject was uniquely assigned an ordered pair of numbers corresponding to age and the fractional number used in his or her tasks. Thus $\underline{m}$:$\underline{n}$ represents an individual $\underline{m}$ months old who was assigned tasks relating to the fraction $\frac{1}{\underline{n}}$. (See Table 5.)

Piaget's conceptual analysis of the fraction concept provides an alternate way of coding and classifying children's responses.

TABLE 5

PERFORMANCE OF SUBJECTS ON FRACTIONS AND CONSERVATION TASKS

Classification \ Task		Area	Length	Set/Subset (2n elements; n recipients)	Set/Subset (4n elements; n recipients)
Fraction	Successful	Jan 67:4 Bes 89:2	Jan 67:4 Sam 91:3 Bes 89:2 Val 66:2	Jan 67:4 Jon 86:3 Len 96:4 Bes 89:2 Val 66:2 Bob 64:3	Sam 91:3 Jon 86:3 Len 96:4 Bes 89:2 Val 66:2
	Transitional	Sam 91:3 Jon 86:3 Len 96:4 Val 66:2	Jon 86:3 Len 96:4 Mic 79:5 Jed 74:4		Mic 79:5 Bob 64:3 Jed 74:4
	Unsuccessful	Bob 64:3	Bob 64:3		
Conservation	Successful	Jan 67:4 Len 96:4 Bes 89:2 Mic 79:5	Jan 67:4 Sam 91:3 Len 96:4 Bes 89:2 Mic 79:5 Jed 74:4	Jan 67:4 Jon 86:3 Len 96:4 Bes 89:2 Val 66:2 Bob 64:3	Sam 91:3 Len 96:4 Bes 89:2 Mic 79:5 Jed 74:4
	Unsuccessful	Bob 64:3 Val 66:2	Bob 64:3 Val 66:2		Jon 86:3 Val 66:2

As mentioned previously, Piaget et al. (1960) identified seven fraction components or subconcepts which unite to form a single mental structure. From this structural unit emerges an anticipatory scheme which guides the child's action throughout the problem-solving process. It is this anticipatory scheme which Piaget believes to be the crucial mechanism in the development of the fraction concept.

It was found in this study that the seven subconcepts identified by Piaget and associates provided an appropriate classification scheme for continuous-dimension tasks (length and area) but not for discrete-dimension tasks. The presence or absence of an understanding of these subconcepts was not easily discerned in the discrete case. (This issue will be considered in more detail later.) Consequently, only the responses to the continuous-dimension tasks will be classified using this scheme. Furthermore, three of the subconcepts were found to be inappropriate for the present data. All subjects in this study understood that a whole is divisible into parts (a very primitive notion) and that there is a fixed relation between the number of parts and the number of divisions (also a primitive notion according to Piaget). Therefore, these two subconcepts provided no discriminatory power and were not used. Since this study did not investigate the relationship between fractional and classification abilities the nesting or hierarchical character of fractional parts is not a relevant subconcept and was not used in the present classification scheme. The remaining four subconcepts appear in Table 6. The notion that subdivision must be exhaustive was made more explicit here by distinguishing between those who considered this an initial requirement and those who achieved it only eventually. As in Table 2, the entry n:m indicates a subject m months old who was assigned tasks relating to the fraction $\frac{1}{n}$. (See Table 6.)

Employing a post hoc classification scheme, responses to the discrete-dimension tasks were categorized according to the amount of attention paid to the part-whole relationship as evidenced by the partitioning strategy employed. This was considered to be in keeping with Piaget's emphasis (in the continuous case) on attention to the whole while subdividing the whole into an appropriate number of equal pieces. Table 7 shows responses categorized in a dichotomous fashion along this dimension. The entry m:n represents an individual m months old who was assigned tasks relating to the fraction $\frac{1}{n}$. Not all subjects were administered both discrete quantity tasks due to time limitations. (See Table 7.)

This classification scheme required some subjective judgements. In particular it should be noted that those responses in the "simple partitioning" category do not necessarily reflect an inattention to the whole. It only indicates that such attention, if it existed, was not clearly evident.

INTERPRETATIONS

The small sample size precludes drawing conclusions from the data. However the results do provide suggestive evidence concerning several of the questions to be investigated in this study. Order of task presentation may affect the strategies employed in the area task.

TABLE 6

FRACTION SUBCONCEPT KNOWLEDGE DEMONSTRATED BY SUBJECTS

Task / Subconcept	Area		Length	
	Yes	No	Yes	No
a. Determinate number of parts (one piece for each recipient)	Jan 67:4 Sam 91:3 Jon 86:3 Len 96:4 Bes 89:2	Bob 64:3 Mic 79:5 Val 66:2	Jan 67:4 Sam 91:3 Jon 86:3 Len 96:4 Bes 89:2 Mic 79:5 Jed 74:4	Bob 64:3
b. Exhaustive subdivision: I = initial exhaustion of whole; E = eventual exhaustion of whole	Jan(I) 67:4 Sam(I) 91:3 Bes(I) 89:2 Bob(E) 64:3 Mic(E) 79:5 Jon(E) 86:3 Len(E) 96:4 Val(E) 66:2		Jan(I) 67:4 Sam(I) 91:3 Jon(I) 86:3 Len(I) 96:4 Bes(I) 89:2 Val(I) 66:2 Jed(I) 74:4 Bob(E) 64:3 Mic(E) 79:5	
c. Equal size pieces	Jan 67:4 Sam 91:3 Jon 86:3 Len 96:4 Bes 89:2 Mic 79:5	Bob 64:3 Val 66:2	Jan 67:4 Sam 91:3 Jon 86:3 Len 96:4 Bes 89:2 Mic 79:5 Val 66:2	Bob 64:3 Jed 74:4
d. Conservation	Jan 67:4 Mic 79:5 Len 96:4 Bes 89:2	Bob 64:3 Val 66:2	Jan 67:4 Sam 91:3 Jon 86:3 Len 96:4 Bes 89:2 Mic 79:5 Jed 74:4	Bob 64:3 Val 66:2

TABLE 7

SUBDIVISION STRATEGIES EMPLOYED BY SUBJECTS

Task / Strategy	Set/Subset 2n elements; n recipients	Set/Subset 4n elements; n recipients
Simple Partitioning (without regard for whole or eventual outcome--usually a one-by-one sharing out procedure)	Bob 64:3 Jan 67:4	Jon 86:3 Len 96:4 Val 66:2 Bob 64:3 Jed 74:4
Consideration of the Whole (initial attention to the whole and to the eventual outcome of the subdivision)	Len 96:4 Jon 86:3 Val 66:2 Bes 89:2	Bes 89:2 Sam 91:3 Mic 79:5

All five children who were given the area-length task order used a pie- or wedge-shaped cutting strategy while two of the four children who received the length-area task order used a vertical or linear cutting strategy. The techniques used in subdividing length may have suggested a strategy for subdividing the circular area which was difficult to correctly carry out. The uniform success and consistent strategies used in the discrete case concealed any order effects which may have existed between the larger number and smaller number tasks.

Concerning the relative difficulty of the three interpretations of fraction used here (area, length, and set/subset) the results suggest that the set/subset discrete case is signficantly easier than the continuous cases of length and area, with the length task slightly easier than the area task. The most reasonable explanation of this order of difficulty seems to be the following. Complete and immediate solutions to the continuous quantity tasks require well developed anticipatory schemes; the discrete quantity tasks do not require such schemes for solution. Subdivision of continuous wholes into a correct number of equal pieces requires an ability to foresee the outcome of this subdivision. In essence this anticipation in the length and area tasks represents an additional prerequisite for solution not found in the set/subset tasks.

This issue can be viewed from a slightly different perspective. The predominant strategy used by the younger children in this study was a counting strategy. Equality of subdivisions was often determined by the number of pieces in each plate. The set/subset tasks were immediately solvable on the basis of number strategies, but the length and area tasks required a subdivision into equal pieces, a step treated by some children as an obstacle to their eventual number solutions.

The relative difficulty of the circular area task, compared to the length task, can possibly be attributed to the more difficult measurement properties of a circle compared with a length. It is simply more difficult to cut a circle into three equal pieces than it is to cut a short string into three equal pieces. This fact was noted by Piaget et al. (1960).

The relative difficulty of fractional numbers must be considered separately for each interpretation. Piaget et al. (1960) suggested that the order of difficulty for the area interpretation was halves, fourths, thirds, fifths, and sixths. Problems involving fourths were mastered prior to those involving thirds because the former could be solved by successive dichotomy, a strategy considered "natural" by Piaget. The results here provide support for this order of difficulty.

The length interpretation appears to produce a slightly different order of difficulty. Since successive dichotomy was used in this task only once, the level of difficulty seemed to correspond to the number of parts, i.e., halves, thirds, fourths, fifths.

The set/subset interpretation produced significantly different results. All tasks were completed successfully; the larger fractional numbers caused little if any additional difficulty. The usual one-by-one partitioning strategies were used as successfully with fifths as with thirds. Halves may provide a slightly easier situation since some children seemed to intuit the solution in this case without utilizing a sustained strategy. However, the other fractional numbers proved to be equally difficult.

The general strategies used to solve the discrete quantity tasks have already been discussed in the results section of this report (see Table 7). It should be noted that children's tendency to consider the whole in these tasks appears to be a function of the number of elements in the initial set. The task with 2n elements and those with halves elicited attention to the whole more frequently than the other tasks.

The strategies used to solve the continuous quantity tasks can be deduced from the results reported in Table . The presence of subconcepts a, b(I), and c implies the existence of an anticipatory scheme and an immediate solution to the task. A combination of subconcepts a, b(E), and c suggest a subdivision of equal pieces without regard for the whole but with regard for the number of pieces. Subjects in this category subdivided a portion of the whole into the correct number of equal pieces, but then had to subdivide the remainder (in an iterative fashion) in order to use up the whole. Most subjects judged the equality of subdivisions by counting the number of pieces in each partition.

Subjects who evidenced only an understanding of subconcept b(E) are those whose strategy consisted of simple fragmentation. No real concern was shown for the size of the pieces, and the number of pieces far exceeded the number of recipients. Piaget et al. (1960) consider this to be the most primitive form of subdivision. Other combinations of subconcepts can be analyzed in a similar way to identify the strategy used to solve the task. Those mentioned above were the most frequently occuring ones in this study.

In general, failure to initially solve the task was due to an incomplete understanding of part-whole relationships rather than part-part relationships. Many children did not initially attend to the whole. They began immediately to cut equal pieces without considering how many of these size pieces would be contained in the whole. In fact, the younger children were concerned only with comparing the size of each piece with the one preceding it to ensure equality, but did not consider how many pieces were being cut or whether the whole would be exhausted. It is not clear whether the child's concern with the size of the pieces, to the exclusion of a concern with part-whole relationships, is a function of their incomplete fraction concept or their social experience. It is probably the case that in their previous experiences with pies and similar items, the pieces are always cut equal but the whole is seldom used up intitially. This no doubt affects their strategy in these tasks.

The relationship between conservation tasks and fraction tasks seems to be that failure on one task indicates failure on the other task. For the continuous quantity interpretations of area and length, three of the four cases of failure on a conservation task were paired with cases of failure on the corresponding fraction task, and three of the four cases of failure on a fraction task were paired with cases of failure on the corresponding conservation task. For the discrete quantity tasks, however, no one was considered to be unsuccessful on the fraction tasks, whereas two were unsuccessful on the conservation tasks. As noted previously, though, success in the discrete quantity fraction tasks may not have always been an indication of an attention to the whole which is necessary for the conservation tasks.

In conclusion, the primary concern of this study was a determination of the conceptual understanding of fraction which children possess with respect to three different physical interpretations--area, length, and set/subset--within the context of constructing fractional parts. It is now recognized that a direct comparison of these interpretations is extremely difficult if not impossible. Piaget et al. (1960) claim that an anticipatory scheme is essential for the development of the fraction concept. This study found that such a scheme is important in the continuous quantity tasks, but it apparently is not needed for successful solution of the discrete quantity tasks. The problem now becomes one of identifying criteria which define a complete fraction concept. It may be necessary to outline a separate set of criteria for the various interpretations of fraction. The seven subconcepts and the resulting anticipatory scheme identified by Piaget et al. (1960) seem to be appropriate only for continuous quantity interpretations. The set/subset discrete case may require an alternate conceptual analysis and a different set of subconcepts or defining criteria.

LIMITATIONS

The fraction concept is complex. Many years of development and learning are required to move from an initial primitive understanding of part-whole and part-part relationships to a functionally mature understanding of symbolic computational procedures. This study has considered only one small aspect of this "fraction concept." In addition, this study dealt only with constructing fractional parts. The recognition of fractional parts, particularly in the continuous quantity case, would no doubt produce significantly different results.

Because of these limitations, this study cannot be directly related to most school situations. For example, it is not clear that the anticipatory scheme, which is essential for an immediate solution to the continuous quantity construction tasks, is necessary for classroom instructional lessons in fractions which usually require only a recognition ability. However it does seem clear that the kind of understanding of fraction assessed here would make conventional fraction instruction more meaningful for the child.

SUGGESTIONS FOR FUTURE RESEARCH

Several comments concerning methodology are in order. First, the clinical interview situation employed in this study is imperative for future investigations of similar questions. Support for this claim comes from a comparison of Bob and Sam in the area task. Neither were successful in completing the task but one demonstrated the most primitive strategy of those observed in the study while the other demonstrated the most sophisticated. It is the process of solution, not the product, which provides the best evidence for a child's level of understanding.

The particular situation employed here, with stuffed animals eating at a party, was well received by the children and can be recommended for future investigations. The materials employed (play-dough pies, licorice strings, and penny-candy) also worked quite well.

Several modifications of the procedure can be suggested. The conservation question poses several problems. Many times the child's solution to a task requires several minutes. To ask a question at the conclusion of this process about the equality of the pieces taken together and the original amount may place too much dependence on the child's memory. In an attempt to minimize this potential problem, the procedure used here was to produce another whole, like the original, at the time of the question. However this was not completely satisfactory since some children seemed to think we were asking a perceptual question, and others appeared to parrot back the response that both quantities were the same (the experimenter informed the child that the second object was "just like the one we started with"). A possible resolution of this problem would be to begin the task with two equal wholes, only one of which is subdivided. The second one could remain in view during the task and serve as the comparison object for the conservation question.

A second suggested procedural modification concerns the follow-up question in the continuous quantity tasks. Many subjects completed their initial subdivision without exhausting the whole. At this point they can either be requested to share out the remainder, or they can be provided with another whole and asked to "use it all up this time." Both of these alternatives were used in this study, but they were not used consistently. In order to compare responses on this task across subjects future studies should select one of these follow-up procedures and use it consistently.

The age range of the sample should be increased. Possibly a range of from four years to eight or nine years would be adequate. The sample should include children who, with respect to area, fail the halving task and children who successfully complete the thirds or fifths task. Such a sample is needed to follow the developmental pattern outlined by Piaget et al. (1960).

The interview time allowed for each subject should be increased several minutes from the 15 minutes per subject used here. Not all subjects require this much time to complete the tasks, but some require much more. A 15 minute average is not quite sufficient.

A final suggestion for future research is of a more substantive nature. A comparison of continuous quantity and discrete quantity interpretations of fraction requires a common classification scheme. This scheme must be more than simple success versus failure on the task. It must be of the type proposed by Piaget et al. (1960), but it must be appropriate for the discrete case as well as the continuous case. Therefore, it appears that some theoretical work involving a conceptual analysis of fraction must precede (logical analysis) and accompany (psychological analysis) future investigations if this comparison is to be meaningful.

PILOT STUDY 4

A PILOT STUDY OF COMPARING-AND-ORDERING-FRACTIONS TASKS WITH FIRST AND SECOND GRADERS

by

William H. Kraus

PURPOSE

The purpose of this pilot study was to investigate the feasibility of measuring the ability of young children (first graders) to compare and order fractional parts without the use of symbolic fraction notation or the use of fraction terminology.

It seems possible that much of the difficulty that children (and adults) have in comparing and ordering fractions results from the use of symbols and terminology and not from a lack of understanding of the underlying concepts. In order to test this hypothesis it is necessary to develop tasks that measure the basic concepts of comparing and ordering fractions without relying on the subject's familiarity with symbolic fraction notation or terminology. A tentative set of such tasks was pilot tested in this study.

A set of four tasks based on an area interpretation of fraction was developed by systematic variation of the three basic elements of a fraction: the unit, the numerator, and the denominator. Thus, in one task the fractional parts to be compared used the same unit, same numerator, and the same denominator; in the other three tasks one of the elements was varied while the other two were held constant. Each of the latter three tasks consisted of two subtasks, so that results would not be confounded by the subject's attention to perceptual clues. A conservation-of-area task was added, since it was thought that the subject's ability to conserve might have some effect on his or her performance on the task.

PROCEDURE

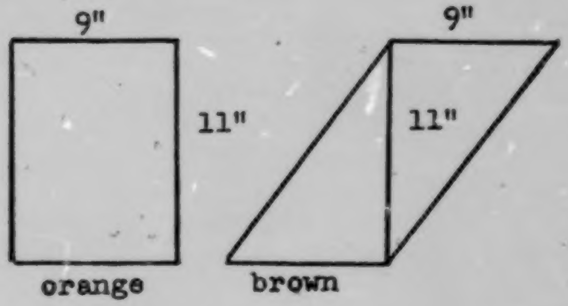
The procedure for administering each of the tasks was similar. The subject watched while two pieces of construction paper (one orange and one brown) were cut in the fashion determined for that task. The pieces were laid in front of the subject in such a way as to approximate the size and shape of the piece of paper before it had been cut. The subject was then reminded that the two original pieces of paper had been the same size (or, in one task, that one had been bigger than the other). The subject was next reminded of the number of pieces into which each original piece had been cut. The fractional parts of interest in the task were then separated from the others, and the subject was asked, "Is there more orange paper, more brown paper, or are they both the same?" After the answer the subject was asked why he or she answered the way he or she did. The order of task presentation and color of paper used were systematically varied from one interview to another.

The subjects were three first graders and one second grader from a parochial elementary school in Madison, Wisconsin. The interviews lasted approximately 15 minutes each and were videotaped.

The rest of this paper will describe each task and discuss subject's responses. Appendix A contains comments on a study by Mary Ann Konsin that attempted to develop tasks to measure the ability of older children (fifth graders) to compare and order fractions.

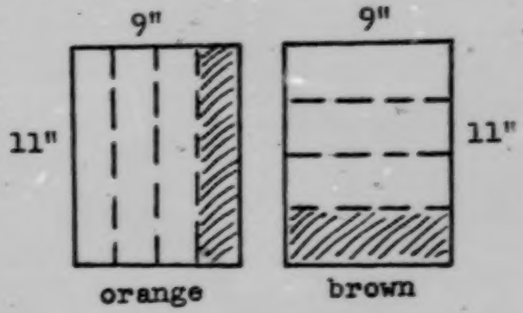
The four subjects (A = the second grader, B, C, and D = the first graders) were given the tasks in the following order (numbers refer to task numbers given below): A and D--3b, 3a, 2, 4a, 4b, 1, 5a, 5b; B and C--the reverse of the above order. The color of the two pieces of paper in each task for A and C was the reverse of the colors for B and D. (Colors used in the diagrams and discussion below are for convenience to the reader. To get the actual colors and responses for A and C "orange" would have to be substituted for "brown" and vice versa.)

TASK ONE--CONSERVATION TASK



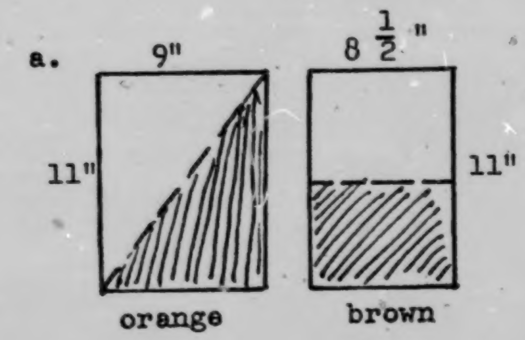
This task began with two pieces of paper the same size and shape. The brown piece was cut diagonally in half and the pieces rearranged to form the above parallelogram. When asked, "Is there more orange paper, more brown paper, or are they both the same?" and then, "Why?" (these same questions were used in all tasks), the subjects gave the following responses: A--the same, because they both look big; B--the same, because they started the same, you just cut one in half; C--brown, because there's two pieces here (pointing to brown) and one piece here (pointing to orange); D--the same, because you could do this (she rearranged the brown pieces to form the original rectangle). B and D gave "good" conservation reasons for their answers and A might have been able to if pressed. C apparently used a number interpretation of "more paper" rather than an area interpretation or possibly used a counting strategy in an attempt to determine the greater area. More data would be needed to compare this task with the fraction tasks below, but the study shows that the use of this task with first graders is feasible.

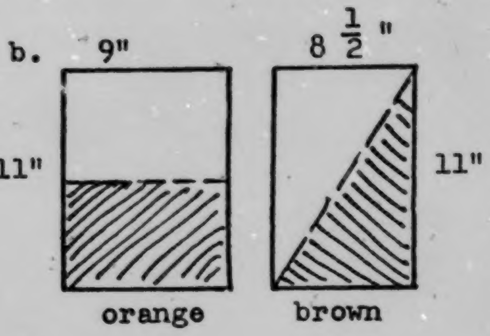
TASK TWO--SAME UNIT, SAME NUMERATOR, SAME DENOMINATOR



The orange piece of paper was cut lengthwise into fourths and the brown piece widthwise into fourths. The subject was asked to compare one piece of each. Responses: A--orange, because it looks longer; B--orange, because it's more bigger; C--orange, because it's skinnier but longer; D--orange (reason not available since the technician had forgotten to turn the videotape recorder on until after this task). All subjects centered on the perceptual cue of length. It might be that the use of fraction terminology would have been a help (rather than the hypothesized hindrance) in the subject's seeing that both parts were equal, e.g., "Is one-fourth of the orange paper more, less, or the same as one-fourth of the brown paper?" might have elicited correct responses from some of the subjects. It is interesting to note that subjects did not pay any attention to the number of pieces into which the paper had been cut when trying to answer questions for any of the tasks.

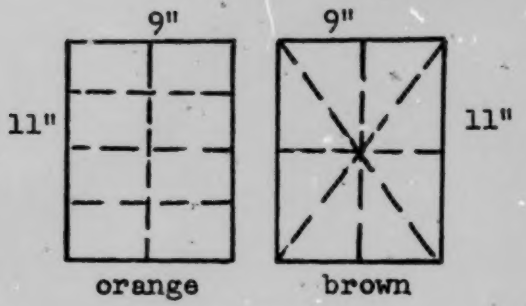
TASK THREE--DIFFERENT UNIT, SAME NUMERATOR, SAME DENOMINATOR

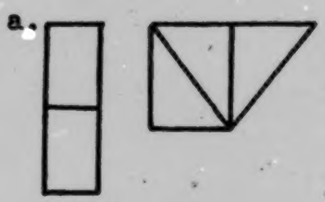


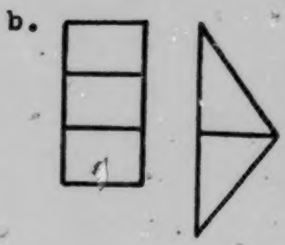


In this task the orange piece of paper was slightly wider than the brown piece. They were cut in half in two ways (see illustration). The subject was to compare one-half of the orange piece with one-half of the brown piece. In Task 3a the perceptual cue favored the correct response, and in Task 3b the perceptual cue favored the incorrect response. Responses to Task 3a: A--orange, because of the way it's cut, there is more of a half here (pointing to orange) than here (pointing to brown); B--orange, because it's bigger; C--orange, because it was a bigger piece to start with; D--either orange or the same, can't tell for sure, the orange is longer, the brown is fatter. C's response may sound encouraging, but her reasoning didn't hold up in the face of the conflicting perceptual cue given in Task 3b. Responses to Task 3b: A—brown, because of the way it's cut, it looks bigger; B--brown, because it's more bigger; C--brown, because the orange is a little wider, but the brown is longer; D--brown or the same, can't tell for sure. Although they knew that the original orange piece of paper had been bigger than the brown, all four subjects based their answers on perceptual cues.

TASK FOUR--SAME UNIT, DIFFERENT NUMERATOR, SAME DENOMINATOR

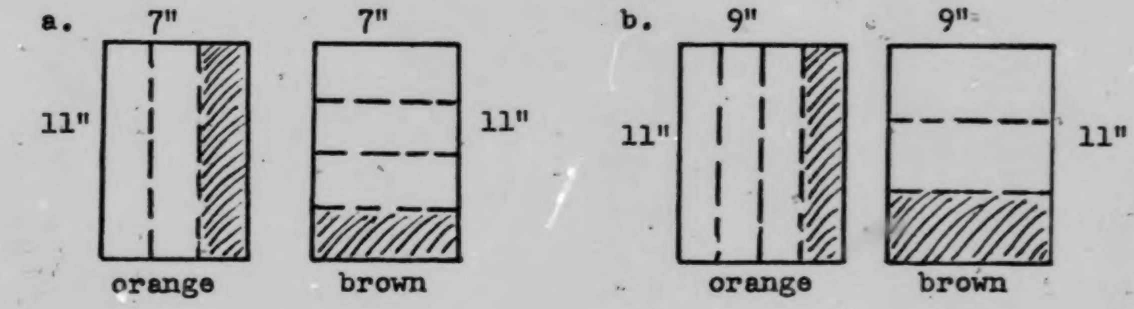






Both pieces of paper were cut into eighths (see illustration). In Task 4a two orange pieces were compared with three brown pieces, and in Task 4b three orange pieces were compared with two brown pieces. Responses to Task 4a: A--brown, because there are more pieces and they are cut bigger; B--brown, because there are more pieces; C--brown, because the orange pieces are smaller but you took more brown ones away (from the original pieces); D--brown, because there are three pieces here (pointing to brown) and two pieces here (pointing to orange). Responses to Task 4b: A--orange, because there are more; B--orange, because there are more pieces; C--orange, because the orange pieces are littler and pieces that are littler are more; D--the same, because the brown pieces are bigger than the orange pieces. Although three out of the four subjects gave correct responses to this task, it is apparent that most did not recognize that the orange pieces were the same size as the brown pieces. C's response to Task 4b is particularly interesting; it may have been caused by some misapplication of the knowledge that for any given unit the smaller you make the sub-units, the more sub-units you will have. This task appears to be the easiest of the four fraction tasks, since application of a counting strategy will give correct results.

TASK FIVE--SAME UNIT, SAME NUMERATOR, DIFFERENT DENOMINATOR



In this task one piece of paper was cut into thirds and the other into fourths. The orange piece was cut lengthwise and the brown piece widthwise (see illustration). The subject was asked to compare one-third of one piece with one-fourth of the other (as with all other tasks this fraction terminology was not used with the subjects). Responses to Task 5a: A--orange, because it's longer; B--the same, not sure why; C--orange, because it's longer; D--orange, because even though the brown is fatter, the orange is longer. Responses to Task 5b: A--brown, because it's fatter (she indicated through her actions and comments that the brown piece was "fatter" to compensate for the orange piece being longer); B--orange, because it's longer; C--the same, because they started the same, you just cut one bigger and one littler; D--the same, because the brown is really fat. Again on this task the perceptual cues dominated; in this case, there was considerable indication that the subjects felt that some degree of extra width in one piece could compensate for extra length in the other.

SUMMARY

In summary, the subjects correctly responded to the following tasks (correctness not based on reason given for response, Tasks 3, 4, and 5 considered correct only if both subtasks were correctly answered): A--Tasks 1, 4, and 5; B--Tasks 1 and 4; C--Task 4; D--Task 1.

In general, the subjects did not pay attention to the unit or the denominator of the fractional parts, but did attend to the numerator. It would be interesting to see at what age Tasks 2, 3, and 5 become readily solvable and whether formal instruction in fractions would have a significant effect on subjects' ability to solve them. (These three tasks were solved by none of the first graders in the study. The second grader solved Task 5. My second grade daughter solved Tasks 2 and 3, but missed Task 5 in a pretrial of the tasks.)

The tasks themselves seem fairly easy to administer and appear to give valid results. The color of the paper did not appear to have any effect, but since responses to one task or subtask seemed to be affected on several occasions by earlier tasks or subtasks, the order of presentation would seem to make a difference.

The research most strongly indicated as potentially useful and interesting is a cross-sectional status study involving the five tasks from this pilot study. The range of such a study might be kindergarten through fourth or fifth grade.

With the higher grades--where most subjects would have had some formal instruction in fractions--videotape analysis of what effect formalized knowledge of fraction concepts would have on the task solutions would seem to be useful in interpreting the data.

APPENDIX A

Mary Ann Konsin carried out a study (concurrent with mine) of fraction concepts of fifth graders. Since there was some degree of similarity between the tasks employed in her study and those employed in mine, some comment seems appropriate.

The majority of the tasks in the Konsin study varied two of the three fraction variables (unit, numerator, and denominator), thus these tasks were theoretically more difficult. There were, however, two tasks used in the study that paralleled tasks in my study; in one unit, numerator and denominator were all held constant, and in the other, the numerator and denominator were held constant while the unit was varied. All four of her subjects solved both tasks correctly, whole none of the subjects in my study solved either of the two similar tasks.

This would seem to suggest that considerable learning of fraction concepts takes place between first and fifth grades. On closer examination, however, such a conclusion is not possible from these two studies. First, the samples were both small and were not comparable (three girls and one boy from a parochial school versus four boys from a public school). Second, the tasks were really not comparable: in the Konsin study the perceptual clues favored the correct answers on both tasks. Finally, in the Konsin study subjects were encouraged to explicitly use fractions in the tasks, while in my study fractions were never mentioned. The Konsin study suggests that fifth graders may be able to solve the more difficult tasks from my study, but the only way to verify this would be to actually use the tasks from my study with a sample of fifth graders.

PILOT STUDY 5

FRACTIONAL PARTS AND FIFTH GRADERS: THE EFFECT OF VISUAL STIMULI ON COMPARING FRACTIONAL PARTS OF THE SAME OR DIFFERENT OBJECTS

by

Mary Ann Konsin

PURPOSE

The purpose of this study was to determine whether fifth graders understand the concept of fraction when visual support is used for various comparisons of fractional parts of wholes. The focus is on comparing fractional parts of figures and determining when such comparisons are meaningful.

RATIONALE

Fractions are considered by many prospective elementary teachers as the stumbling blocks to success in teaching mathematics in the elementary school. A short survey conducted by this investigator provided this information. Feeling insecure of their own performance on computation tasks with fractions provides additional anxiety when they are faced with introducing fraction concepts to their students. Most have memorized rules governing the mathematical operations of fractions and have personally left the concept at that point. Novillis (1976) addressed the complex nature of the fraction concept and provided a set of hierarchically linked subconcepts and examples of specific behaviors related to the subconcepts.

With teacher apprehensiveness and the complexity of the fraction concept evident, it is no wonder that suggestions regarding the placement of this topic in the school curriculum have included the secondary school. The assumption seemed to be that older students would be able to achieve a functional rational number concept with a limited amount of instruction. An alternative to postponement of the topic is the development of better instruction for elementary school children. The direction for that development can come from empirical data indicating the ability of students to work with symbolic as well as visual representations of fractions and fractional parts.

Kieren, in a working paper, provides five tasks which he feels, if mastered, indicate a mature functioning with rational numbers. One of these, part-whole, has also appeared in Novillis (1976). It has been the basis for both the traditional as well as the modern development of meaning with respect to fraction. Current use has focused on set/subset and shaded regions.

In ordering fractions and comparing fractional parts of the same whole students are usually taught three specific relations. Symbolically these relations are:

a. If $\underline{b}$ is greater than $\underline{c}$, then $\underline{b}/\underline{d}$ is more than $\underline{c}/\underline{d}$.

b. If $\underline{a}$ is greater than $\underline{b}$, then $1/\underline{a}$ is less than $1/\underline{b}$.

c. If $\underline{a}$ is greater than $\underline{b}$, then $\underline{c}/\underline{a}$ is less than $\underline{c}/\underline{b}$.

A fourth result is presented with respect to comparing fractional parts of different objects.

d. If object $\underline{A}$ is bigger than object $\underline{B}$ with respect to some attribute, then $1/\underline{b}$ of $\underline{A}$ is bigger than $1/\underline{b}$ of $\underline{B}$ with respect to that attribute. Likewise $\underline{a}/\underline{b}$ of $\underline{A}$ is bigger than $\underline{a}/\underline{b}$ of $\underline{B}$.

These basic relations regarding comparisons of fractions provide the foundation for equivalent fractions and combining of fractional parts. The interest here is to consider the effect visual stimuli have on the interpretation of an extension of these relations.

DESIGN

In order to present an extension of the four stated relations a schematic diagram is provided in which each of three variables--unit, numerator, and denominator--assume two values, same or different. The representation is in Figure 3.

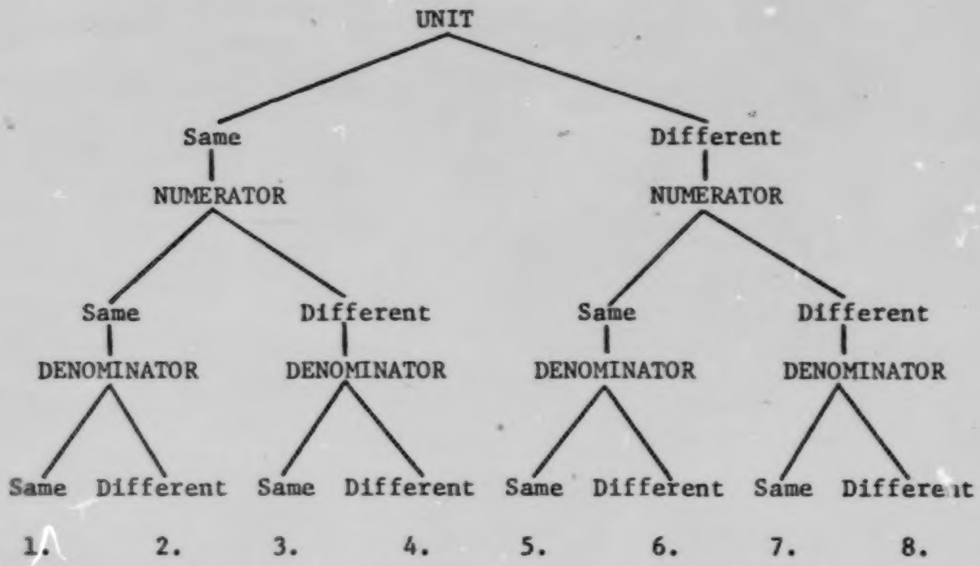

UNIT
Same
Different
NUMERATOR
NUMERATOR
Same
Different
Same
Different
DENOMINATOR
DENOMINATOR
DENOMINATOR
DENOMINATOR
Same
Different
Same
Different
Same
Different
Same
Different
1.
2.
3.
4.
5.
6.
7.
8.

Figure 3: Schematic of tasks when unit, numerator, and denominator take on the same or different value.

This investigation is working with Tasks 1, 4, 5, 6, 7, and 8 from the schematic. The question here is what effect visual stimuli have on the ability of fifth grade students, who have finished formal symbolic instruction in fractions, to order and compare fractional parts of the same or different objects. The interpretation of these tasks and the questions asked in the videotaped interview follow. The actual task drawings used in the study are found in Appendix A. Each figure was on a separate sheet of paper to facilitate handling by the student. The numbers in parentheses indicate the task numbers in the study.

1. (3) Same Unit; Same Numerator; Same Denominator

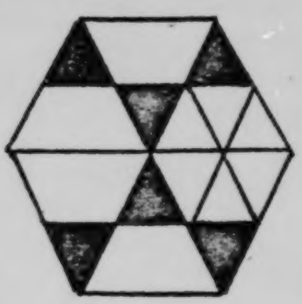


Figure 1

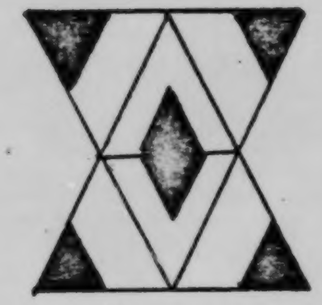


Figure 2

The interpretation of this task is an example of equivalent fractions where the visual stimulus is a complex figure with some of the lines omitted.

Questions:

A. Are the number of shaded regions in Figure 1 the same as the number of shaded regions in Figure 2?

If Yes: Is the area in Figure 1 the same as the area in Figure 2?

If No: Explain.

B. What fraction is reprsented by the shaded parts in Figure 1?

What fraction is represented by the shaded parts in Figure 2?

4. (1) Same Unit; Different Numerator; Different Denominator

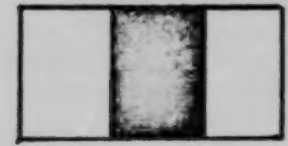


Figure 1

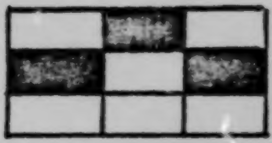


Figure 2

The interpretation of this task is an example of equivalent fractions without a complex figure.

Questions:

A. Is the shaded area in the first figure larger or smaller or equal to the shaded area in the second figure? Explain your answer.

B. What fraction is represented by the shaded area in Figure 1?

What fraction is represented by the shaded area in Figure 2?

C. How does (answer B,1) of Figure 1 compare to (answer B,2) of Figure 2? Explain.

5. (2) Different Unit; Same Numerator; Same Denominator

a. With respect to regions

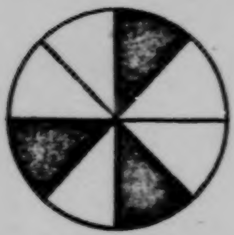


Figure 1

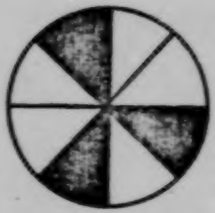


Figure 2

The interpretation of this task is an example of: If object *A* is bigger than object *B* with respect to some attribute then 1/*b* of *A* is bigger than 1/*b* of *B* with respect to than attribute. Likewise *a*/*b* of *A* is bigger than *a*/*b* of *B*.

Questions:

A. Into how many parts is Figure 1 divided?
Into how many parts is Figure 2 divided?

B. How many of the parts are shaded in Figure 1?
How many of the parts are shaded in Figure 2?

C. Is the number of shaded parts in Figure 1 the same as the number of shaded parts in Figure 2?

D. What fraction is represented by the shaded parts in Figure 1?
What fraction is represented by the shaded parts in Figure 2?

E. How do these figures compare?

F. How does (*answer D,1*) of Figure 1 compare to (*answer D,2*) of Figure 2? Explain.

b. With respect to sets

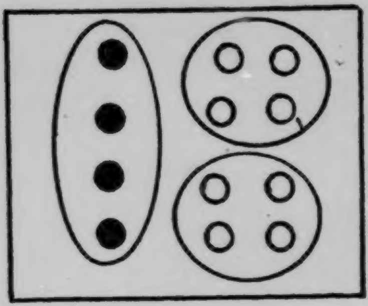


Figure 1

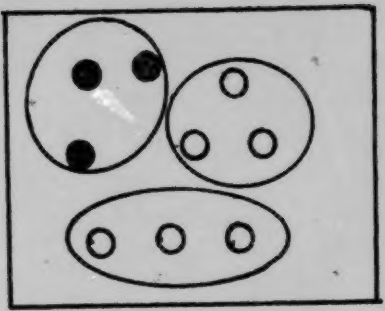


Figure 2

Questions:

A. How many groups are in Figure 1?

How many groups are in Figure 2?

B. How many circles are in each group in Figure 1?

How many circles are in each group in Figure 2?

C. Is the number of shaded circles in Figure 1 the same as the number of shaded circles in Figure 2?

D. What fraction is represented by the shaded circles in Figure 1?

What fraction is represented by the shaded circles in Figure 2?

E. How do these fractions compare?

F. How does (answer D,1) of Figure 1 compare to (answer D,2) of Figure 2? Explain.

6. (5) Different Unit; Same Numerator; Different Denominator

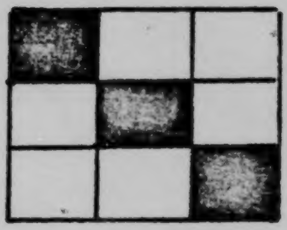


Figure 1

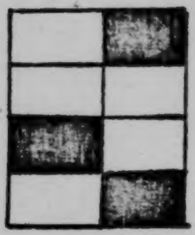


Figure 2

The interpretation of the task is that although the fraction representing the shaded part in Figure 2 is larger than the fraction representing the shaded part in Figure 1, the shaded area in Figure 1 is larger than the shaded area in Figure 2.

Questions:

A. Into how many parts is Figure 1 divided?
Into how many parts is Figure 2 divided?

B. How many parts are shaded in Figure 1?
How many parts are shaded in Figure 2?

C. What fraction of Figure 1 is shaded?
What fraction of Figure 2 is shaded?

D. How do these fractions compare?

E. How does (answer C,1) of Figure 1 compare to (answer C,2) of Figure 2? Explain.

7. (4) Different Unit; Different Numerator; Same Denominator

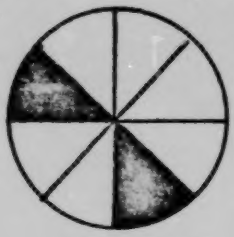


Figure 1

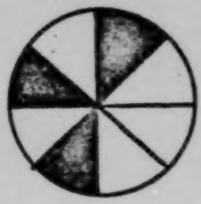


Figure 2

The interpretation of this task is that although the smaller circle has more sections shaded, the area shaded in the larger circle is more than the area shaded in the smaller circle.

Questions:

A. Into how many parts is Figure 1 divided?

Into how many parts is Figure 2 divided?

B. How many parts are shaded in Figure 1?

How many parts are shaded in Figure 2?

C. What fraction of Figure 1 is shaded?

What fraction of Figure 2 is shaded?

D. How do these fractions compare?

E. How does (answer C,1) of Figure 1 compare to (answer C,2) of Figure 2?

8. (6) Different Unit; Different Numerator; Different Denominator

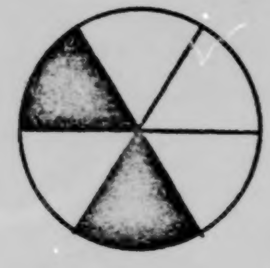


Figure 1

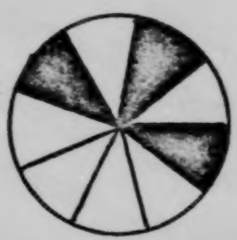


Figure 2

The interpretation of this task is that although the two fractions are equal and the smaller figure has more parts shaded, the shaded area in Figure 2 is larger than the shaded area in Figure 1.

Questions:

A. Into how many parts is Figure 1 divided?
   Into how many parts is Figure 2 divided?

B. How many parts are shaded in Figure 1?
   How many parts are shaded in Figure 2?

C. What fraction of Figure 1 is shaded?
   What fraction of Figure 2 is shaded?

D. How do these fractions compare?

E. How does (answer C,1) of Figure 1 compare to (answer C,2) of Figure 2? Explain.

PROCEDURE

The investigator interviewed four fifth grade male students at the Instructional Media Distribution Center at the University of Wisconsin-Madison. The first two interviews took place on May 12, 1977 and the last two on May 16, 1977. All of the interviews were videotaped. The students performed the six tasks in an order determined by picking an identifying number from an envelope.

RESULTS

The results of the investigation will be presented with respect to each task.

1. Same Unit: Different Numerator: Different Denominator

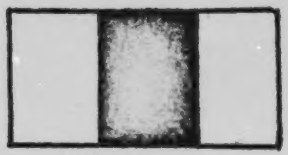


Figure 1

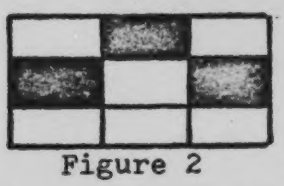


Figure 2

The responses to this item are presented in Table 8. Each subject determined correctly that the shaded areas in both figures were the same. In his explanation each subject moved one paper over the other to show that the total area is the same and then indicated how each of the smaller rectangles could be moved to correspond to the larger rectangle in Figure 1. The four subjects "divided down" three-ninths to one-third easily determining that the fractions were equivalent.

TABLE 8

SUBJECT AND CORRECT RESPONSES TO QUESTIONS FOR TASK 1

Item	Subjects				
	1	2	3	4	Correct
Comparison of Shaded Areas	Same	Same	Same	Same	Same
Fractions Represented					
Figure 1	1/3	1/3	1/3	1/3	1/3
Figure 2	1/3	3/9 or 1/3	3/9	3/9	3/9
Comparison of Fractional Parts	Same	Same	Same	Same	Same

2. Different Unit: Same Numerator: Same Denominator

a. With respect to regions

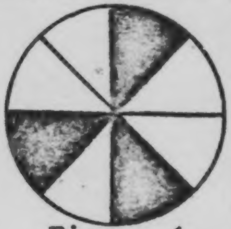


Figure 1

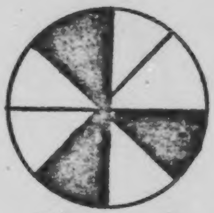


Figure 2

b. With respect to sets

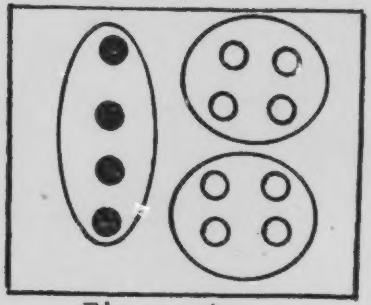


Figure 1

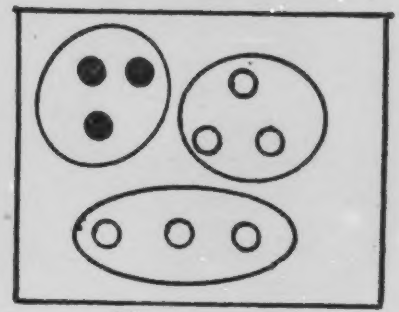


Figure 2

The responses to this item are presented in Table 9. Subject one, when comparing the fractional parts of the regions first indicated that the areas were the same but then in the course of explaining why, changed his answer since one circle was larger than the other. He then concluded that the area shaded in Figure 1 was more than the area shaded in Figure 2. Each subject replied to the questions correctly, although subject four required a greater length of time in determining the response to each of the questions regarding the comparing of fractional parts.

TABLE 9

SUBJECT AND CORRECT RESPONSE TO QUESTIONS FOR TASK 2: REGIONS AND SETS

Item	Subjects				
	1	2	3	4	Correct
a. Regions					
Parts Divided					
Figure 1	8	8	8	8	8
Figure 2	8	8	8	8	8
Parts Shaded					
Figure 1	3	3	3	3	3
Figure 2	3	3	3	3	3
Comparison of Shaded Parts	Same	Same	Same	Same	Same
Fraction Represented					
Figure 1	3/8	3/8	3/8	3/8	3/8
Figure 2	3/8	3/8	3/8	3/8	3/8
Comparison of Fractions	Same	Same	Same	Same	Same
Comparison of Fractional Parts	3/8 of Fig. 1 bigger	3/8 of Fig. 1 bigger	3/8 of Fig. 1 bigger	3/8 of Fig. 2 smaller	3/8 of Fig. 1 bigger
b. Sets					
Groups					
Figure 1	3	3	3	3	3
Figure 2	3	3	3	3	3
Circles in Group					
Figure 1	1[a]	4	4	4	4
Figure 2	1[a]	3	3	3	3

TABLE 9 (CONTINUED)

SUBJECT AND CORRECT RESPONSE TO QUESTIONS FOR TASK 2: REGIONS AND SETS

Item	Subjects				Correct
	1	2	3	4	
Comparison of Shaded Circles	Not Same	Not Same	Not Same	Not Same	Not Same
Fraction Represented					
Figure 1	1/3	4/12	4/12	4/12	1/3
Figure 2	1/3	3/9	3/9	3/9	1/3
Comparison of Fractions	Same	Same	Same	Same	Same
Comparison of Fractional Parts	More circles Fig. 1	Fig. 1 bigger	More in Fig. 1	More in Fig. 1	More in Fig. 1

[a] Question asked of first subject: How many groups are shaded in Figure 1; How many groups are shaded in Figure 2?

3. Same Unit: Same Numerator: Same Denominator

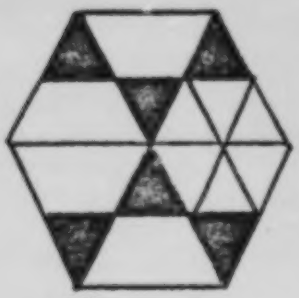


Figure 1

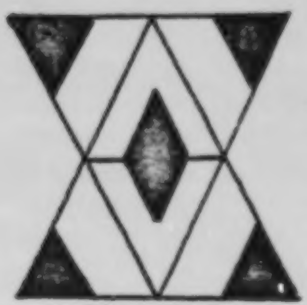


Figure 2

The responses to this item are presented in Table 10. Each subject correctly determined the number of shaded regions in each figure and was able to show that the total areas of both figures were the same. Their method of showing that the areas were the same involved moving part of one figure into a position so that it would become the other figure. Three of the four subjects correctly identified the fractional parts of the figures. Subject three indicated that he was unable to determine the fraction because the parts weren't the same. He then broke the larger triangles of Figure 1 into smaller ones and then determined the correct fraction. Subject four asked a similar question when he indicated that he needed to determine the total number of triangles (small) before determining the fraction. Subject one incorrectly answered the question about what fraction was represented by the shaded parts in Figure 1 and in Figure 2. He counted the regions in each figure and was satisfied with his results. All of the subjects worked first with Figure 1 to determine the fraction. When asked about the fraction represented

TABLE 10

SUBJECT AND CORRECT RESPONSE TO QUESTIONS FOR TASK 3

Item	Subjects				
	1	2	3	4	Correct
Shaded regions the same?	Yes	Yes	Yes	Yes	Yes
Areas the same?	Yes	Yes	Yes	Yes	Yes
(Proof)	(Move one part of either figure to make other figure.)				
Fraction Represented					
Figure 1	6/16	6/24	6/24	6/24	6/24
Figure 2	6/12	6/24	6/24	6/24	6/24

in Figure 2 all subjects, except subject 1, used the fact that the areas were the same in order to determine the second fraction. Each of the subjects discussed the shaded area and all but subject 1 indicated they were the same. Since subject 1 identified the fractions incorrectly he indicated that the shaded area in Figure 2 was larger than the shaded area in Figure 1.

4. Different Unit: Different Numerator: Same Denominator

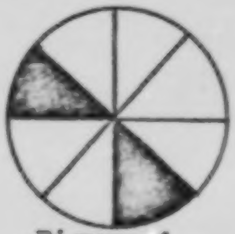


Figure 1

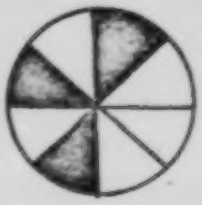


Figure 2

The responses to this item are in Table 11.

TABLE 11

SUBJECT AND CORRECT RESPONSE TO QUESTIONS FOR TASK 4

Item	Subjects				
	1	2	3	4	Correct
Parts divided					
Figure 1	8	8	8	8	8
Figure 2	8	8	8	8	8
Parts Shaded					
Figure 1	2	2	2	2	2
Figure 2	3	3	3	3	3
Fraction Represented					
Figure 1	2/8 or 1/4	2/8	2/8	2/8	2/8
Figure 2	3/8	3/8	3/8	3/8	3/8
Comparison of Fractions	2/8 is smaller	Same denominator 3/8 is bigger	3/8 is bigger	3/8 is bigger	3/8 is greater than 2/8
Comparison of Fractional Parts	Fig. 1 looks bigger	Fig. 1 bigger circle bigger sections	Fig. 1 is bigger looks like 2/8 is bigger	2/8 are bigger than 3/8	2/8 of Fig. 1 has more area than 3/8 of Fig. 2
	Subject 3: Not true because circle is smaller.				

Each subject determined the fractions represented by the shaded parts; made the correct comparison of those fractions and was able to determine that two-eighths of the larger circle represented more area than three-eighths of the smaller circle. Subjects 1 and 3 were not overly comfortable with this result, but they indicated that the area in Figure 1 "looked bigger."

5. Different Unit: Same Numerator: Different Denominator

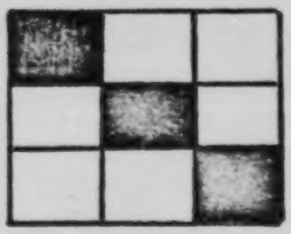


Figure 1

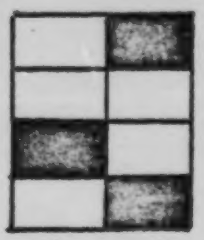


Figure 2

The responses to this item are found in Table 12. Each subject determined the fractions represented by the shaded parts in each figure; three of the four subjects made the correct comparison of the two fractions. One subject (subject 4) attempted to work with information that he knew about eighths. He knew that one-third was larger than one-fourth and smaller than one half. He also knew that two-eighths was the same as one-fourth and four-eighths was the same as one-half, but he was unable to work three-eighths into his understanding of those fractions. Subject 3 indicated he couldn't tell about the fractional parts of the two figures because the squares were not equal in both figures. He felt that although Figure 2 was smaller, the squares in Figure 1 were smaller.

TABLE 12

SUBJECT AND CORRECT RESPONSE TO QUESTIONS FOR TASK 5

Item	Subjects				
	1	2	3	4	Correct
Parts Divided					
Figure 1	9	9	9	9	9
Figure 2	8	8	8	8	8
Parts Shaded					
Figure 1	3	3	3	3	3
Figure 2	3	3	3	3	3
Fraction Represented					
Figure 1	3/9	3/9	3/9	3/9	3/9
Figure 2	3/8	3/8	3/8	3/8	3/8
Comparison of Fractions	3/8 is bigger	3/8 is more	3/8 is bigger	Same	3/8 is greater than 3/9
Comparison of Fractional Parts	Looks like Fig. 1	Not sure Might be same	Can't really tell	Fig. 1 larger than 3/8 in terms of area	Shaded area in Fig. 1 more than shaded area in Fig. 2

6. Different Unit: Different Numerator: Different Denominator

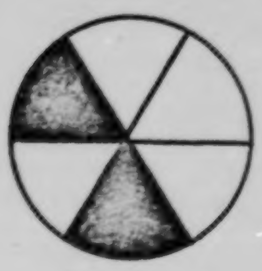


Figure 1

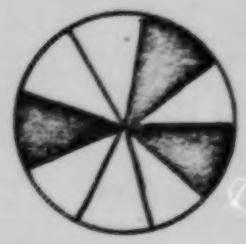


Figure 2

The responses to this item are presented in Table 13. Each subject determined the fractions represented by the shaded area correctly and compared the two fractions correctly. Three of the four subjects correctly indicated that the shaded area of Figure 1 was larger than the shaded area of Figure 2. Only one subject (subject 4) showed that relationship by placing Figure 1 on top of Figure 2. Subject 2 indicated that the shaded area in Figure 2 was more than the shaded area in Figure 1.

INTERPRETATION OF THE RESULTS

The results of this investigation will be interpreted with respect to three areas:

A. Determining the fractions represented by shaded parts of a figure.

B. Comparing fractions represented symbolically.

C. Comparing fractional parts of the same or different figures.

TABLE 13

SUBJECT AND CORRECT RESPONSE TO QUESTIONS FOR TASK 6

Item	Subjects				
	1	2	3	4	Correct
Parts Divided					
Figure 1	6	6	6	6	6
Figure 2	9	9	9	9	9
Parts Shaded					
Figure 1	2	2	2	2	2
Figure 2	3	3	3	3	3
Fraction Represented					
Figure 1	2/6 or 1/3	2/6 or 1/3	2/6	2/6	2/6
Figure 2	3/9 or 1/3	3/9 or 1/3	3/9	3/9	3/9
Comparison of Fractions	Same	Equal	Same (hesitant)	Same	Same
Comparison of Fractional Parts	Fig. 1 bigger	Fig. 2 larger	Fig. 2 smaller	Fig. 1 larger	Shaded area in Fig. 1 is more than shaded area in Fig. 2

A. Determining the Fractions Represented by Shaded Parts of a Figure.

In all of the tasks, all of the subjects except subject 1 in Task 3 were able to determine the fraction represented by the shaded parts correctly. Of interest is the response to this question for Task 2b (sets). Although the investigator focused on groups in each figure with the initial question, three of the four subjects provided the fraction as the number of shaded circles with respect to the total circles rather than the number of groups with respect to the total group. It appeared that the subjects ignored the clue provided in the picture. Possibly the subjects had not worked with the set/subset construct of fraction yet. This information was neither solicited nor offered.

Subject 1 in working with Task 3 felt no concern when determining the fractional part regarding the size of the individual regions. He was very precise when asked to verify six-sixteenths and six-twelfths. With no hesitation he counted the various regions, not worrying if they were the same size at all. The other three subjects indicated the necessity for having equal parts and after determining the total number of small triangles provided the correct response.

B. Comparing Fractions Represented Symbolically

The subjects had no difficulty in comparing fractions with same denominators nor when one fraction could be "divided down" to the other. That situation occurred in five of the six tasks. In Task 5 where three-ninths was to be compared with three-eighths one subject indicated that his lack of knowledge about "eighths" caused him difficulty in terms of answering the question. The other subjects correctly answered that question but the investigator did not require an explanation of their choices.

C. Comparing Fractional Parts of the Same or Different Figures

When the same figure was divided into a different number of parts where the shaded parts represented equivalent fractions (Task 1) the subjects had no difficulty in providing an explanation of why the shaded areas were the same. When a more complex diagram was used for the first figure and the second figure was obtained by rearranging the parts of the first figure (Task 3) the subjects had no trouble showing that the two figures had the same area. However, one subject was unable to correctly identify the fractions represented so he incorrectly indicated that the second figure had a greater shaded area than the first figure.

When the figures were not the same size but did have the same shape (Tasks 2a, 4, 6), in only one case did a subject answer incorrectly regarding which area was greater (Task 6, subject 2).

The task where the figures were neither the same shape nor the same size (Task 5) caused the greatest difficulty for the subjects when they attempted to compare the fractional parts of the two figures. Subject 2 was not sure of the comparison and felt they could be the same. Subject 3 indicated he couldn't really tell but felt that although Figure 2 was smaller, the squares in Figure 1 were smaller.

In summary, it appears that fifth grade boys are able to successfully determine what fractions represent shaded parts of figures, are able to successfully compare these fractions symbolically, but have some difficulty in determining the relation of fractional parts of figures especially when the figures are neither the same shape nor size and divided into a different number of parts.

CONCLUSIONS AND LIMITATIONS OF THE STUDY

The results of this study indicate that fifth grade boys, although currently working with fractions in a symbolic mode, are able to correctly compare fractional parts of figures when the figures have the same shape and size or when the figures have the same shape but different size. This is true when the figures are divided into a different number of parts of the same number of parts. Fifth grade boys have difficulty comparing fractional parts of figures when the figures have different shapes and are not the same size.

A limitation of this study is the sample—fifth grade boys. In two pilots of this study, two girls were interviewed, one in the sixth grade and the other in the fifth grade. The sixth grade student had no difficulty with any of the questions but the fifth grade student showed other areas of difficulty besides the one indicated in this study. Another limitation of this study could be the lack of verification requested and supplied in the course of the interview with respect to comparing fractions. The subjects were never asked to explain why one fraction was larger or smaller than the other. Of interest is Task 5 where the students compared three-eighths and one-third. Were the students influenced in their decision by the diagram for that task, or did they have some method of determining the relation of those two fractions?

DIRECTIONS FOR FURTHER RESEARCH

The large looming question left unanswered in this study is how do the females in grade five respond to these tasks? Is their understanding of the fraction concept at the same level as the boys'? Also, since complex figures do not cause difficulties when students are asked to interpret fractional parts, what would happen if students are presented with a fractional part and asked to provide the whole figure? Will they provide simple or complex figures? Can they build from a part to the whole figure?

SUMMARY OF THE FIVE PILOT STUDIES

In summary, the pilot studies were useful in revealing the good points as well as the weak points of certain techniques of investigation in initial fraction concepts. It was generally felt that the interview situation provided a good means of gaining the information desired in a status study on initial fraction concepts. Blume and Mitchell found that many circumstantial factors (the camera room setting, for example) adversely influenced the children's performance and verbalization in the game. Burke and Schuerger found that while certain tasks may indicate that a person is applying correct criteria for recognizing thirds and fourths, they were of little use in determining whether a child could construct, count, or name fractional parts. They also found that because of complications with children's abilities to determine when two areas were the same, the congruent-parts criteria was hard to assess and new tasks needed to be devised. Hiebert and Tonneson concluded that the Piagetian approach was a good approach for a study of their kind. They also concluded that a framework needed to be developed by which they could describe a child's "part-whole understanding" with respect to discrete quantity partitionings. Kraus found that very few second or first graders could perform the tasks he had developed comparing fractional parts of wholes. This led him to raise the question of whether or not the understandings required in his tasks were developmental and hence, to some degree, independent of formal instruction in fractions, or whether they could be directly taught. The author was satisfied with the adequacy of the interview setting using those tasks. Konsin found the research technique and tasks to be adequate for her purposes but recommended that more time be given to the children's explanation of their answers.

With respect to the status, readiness, and representation questions some hypotheses were suggested. The Hiebert-Tonnesson results seem to suggest that the set/subset representation of fractions may be easier for children in the initial fraction development where notions of equivalence partitionings depend on a numerosity scheme, rather than area measurement and anticipatory schemes. Burke and Schuerger could find no outstanding differences in the children's abilities to work with the set/subset representation and the area representation of fractions. This seemed to be the same for the processes of identifying, naming, and constructing fractional parts of wholes. Hiebert and Tonnesson found the length and region representations to be very much the same.

On the readiness question, Burke and Schuerger concluded that children in second grade can achieve significant progress in naming, identifying, and constructing fractional parts of regions and sets. In light of the fact that development of part-whole understandings and equivalence partitioning abilities vary according to representation, Hiebert and Tonnesson raise the question of whether or not readiness is also dependent on representation. Because of the difference between learning to recognize and learning to construct fractional parts of wholes, they see the need for more work to be done before a study of their sort could be addressed directly to the readiness question.

In regard to the status question, Burke, Schuerger, Mitchell, and Blume all found children to have a common misconception that a third was any subset of three things and a fourth was any subset of four things. Equal-sized parts also seemed to be a common place of error for children. The results did seem to show, however, that children knew that a whole had to be divided into three parts in order to be divided into thirds, and likewise four parts in order to be divided into fourths. Mitchell and Blume, and Hiebert and Tonnesson found that children had some difficulties partitioning regions and sets. Burke and Schuerger found that children in second grade showed poor understanding of the sub-unit and the unit in a fractional situation. Finally, Kraus found that children had not developed well the abilities for using partitioning information in comparing fractional parts of wholes, while Konsin's results seem to show that these abilities do develop to a great extent by grade five. The question is, how much of this development is due to formal instruction in fractions?

VI

OVERVIEW OF DMP APPROACH TO FRACTIONS

DMP (Developing Mathematical Processes) approaches fractions primarily through the part-whole relationship. Children learn that if a whole is divided into equal parts, those parts are fractional parts and can be given special names. For example, each part of a whole divided into three equal parts is called a "third." When the fractional notation is introduced, the children learn that the denominator represents the kind of fractional part the whole is divided into and the numerator represents the number of those parts.

The part-whole development of fractions is done with three variations: (1) the whole considered as a single object (a region, a length, a capacity, etc.); (2) the whole considered as a set of objects; and (3) the whole considered as a unit of measurement (feet, quarts, etc.). When a whole is a set, the parts are taken to be subsets. Thus the situation arises where a fractional part has more than one object in it. When a whole is considered as a unit the parts are treated as sub-units. This interpretation is brought out most clearly in measurement situations. In those situations, something can be "equal" to many of that whole. Mixed numbers (used in measurement situations with repsect to length, capacity, and weight) are instrumental in developing this interpretation of fractions. The key difference between fractions as parts of a whole considered as a single object and fractions as sub-units of a unit is that a fraction such as 13/4 becomes awkward under the first interpretation whereas they are perfectly natural under the second interpretation.

The DMP math program is divided into seven levels, K through 6, with each level consisting of 12 to 15 separate topics. The levels are numbered roughly according to the grade levels in which they might be used. Fractions are introduced at the start of Level 2. Here children are presented first with the whole taken as a single object (a region, a length, or a volume quantity—Topics 28, 31, 34, and 35) and then are introduced to the set representation (Topic 37). These two representations are used concurrently throughout most of DMP.

The unit-sub-unit representation is first clearly brought forward at the end of Level 3 in Topic 50 where mixed numbers are introduced and fractional parts of units are used to measure things. Though the children had previous experience with units and sub-units (eg., quarts and pints) and although they were aware of the fractional relationships that held between those units (1 pint = 1/2 quart) emphasis was not placed on the sub-unit as being a fractional part. (Things were measured to be 10 ft. 6 in. instead of 10-1/2 ft.) The measurement situations in Levels 2 and 3 do, however, provide background for the unit-sub-unit representation. For the most part, in Levels 4, 5, and 6, all three variations (specific object, set, unit) are used in representing fractions.

The part-whole interpretation with its three variations is developed in DMP with respect to the attributes of length, area, numerousness, capacity, and weight. The development of all abstract notions of fractions is preceded by a development of concrete notions underlying the abstractions using the above attributes. The children first learn to recognize, label, represent, and construct fractional parts of wholes considered with respect to those attributes. They learn to measure, compare, and order things on the basis of fractional parts with respect to those attributes. Finally, they are introduced to the various operations dealing with fractions through activities that involve operations with fractional parts of wholes considered with respect to those attributes.

Paying attention to what the whole is and what the sub-unit is in a fractional situation usually is emphasized. The children learn that three-sevenths of one length may be bigger than three-fifths of another length. When taking fractional parts of sets, children are repeatedly reminded that the set is the whole and that fractional parts can have more than one thing in them.

Fractions are ordered by comparing fractional parts of representations provided for the children or constructed by the children. More symbolic methods of ordering and comparing fractions are gradually developed but the reasoning process that is encouraged in DMP reflects the part-whole interpretation. For instance, in ordering three-fifths and three-fourths, children are encouraged to compare the kinds of fractional parts, i.e., fifths and fourths. They are expected to know that the more parts a whole is divided into the smaller those parts will be. As a result, fourths will be larger than fifths and hence three-fourths will be larger than three-fifths. Equivalent fractions are introduced as alternative names for the same fractional part.

Addition and subtraction of fractions is introduced in terms of joining and separating fractional parts of wholes. When this is done in Topic 67, the children by that time are familiar with mixed numbers. Hence the unit-sub-unit interpretation quickly follows the activities of joining and separating by way of introducing mixed numbers into sums and differences. Thus, in addition and subtraction (just as is done ultimately for ordering, comparing, and finding equivalent fractions) the part-whole interpretation is linked with the unit-sub-unit interpretation by considering the whole as a unit. Consistent with the sub-unit interpretation, children learn that when the fractions have the same kind of fractional part they can be added by simply adding the numerators; e.g., in adding 3/7 + 2/7 the children think of joining three-sevenths with two more sevenths to get five-sevenths. When fractions have different fractional parts, the children are led to conclude that they need to change one or both kinds of fractional parts so that the fractions have the same kind of fractional parts (units) before adding or subtracting.

This briefly is the DMP approach to fractions. The following is a synopsis of the various topics in DMP that deal directly with initial fraction concepts. The last topic considered will be Topic 67 at the beginning of Level 5. Further work is done in DMP on fractions but that work presupposes the initial fraction development that we are concerned with here. A brief description of Levels K and 1 and other topics dealing with attribute and measurement prerequisites for the part-whole interpretation of fractions is included.

In Level K, children get much experience working with real objects considering the attributes of length and numerosity. In Topic 2, real objects are directly compared and then ordered on length. The children determine whether two objects are the same length and, if not, determine which is longer. After comparing and ordering two objects or pictures of objects on length the children, in Topic 3, make them equal in length by adding on to the shorter or taking away from the longer. In Topic 4, the children order more than two objects on the basis of length from longest to shortest. The children then learn, in Topic 5, to compare and order lengths indirectly by using a physical representation of one of the lengths.

With respect to numerosity, children learn to compare and order sets on the basis of one-one correspondences (Topic 7). They learn to physically represent the numerousness of a set (Topic 9) and then represent the numerousness with tallying (Topic 12). Finally the children learn the numbers from one to ten and their symbolic representation.

In Level 1, children learn to measure things and continue to consider real objects with respect to the attributes of numerosity and length while introducing the attributes of weight and capacity.

In Topic 16, the children compare and order objects on the basis of weight. In Topic 19, they learn to use arbitrary units to represent lengths or weights of objects by assigning a number or unit. They compare and order objects using these measurements. In Topic 22, the children directly compare and order the capacities of various containers by pouring from one into another. They also learn to represent the capacity of a given container by assigning a measurement with arbitrary units.

Thus, before fractions are introduced in Topic 28, much preparatory work is done in Levels K and 1. More preparatory work for initial fraction concepts is done in Levels 2 and 3. In Level 2, the children learn to separate any set by grouping the objects of the set into subjects of a given size (Topic 30) and to divide a set into a given number of equal subsets and a remainder (Topic 37). As a preparatory objective in Topic 37, the children also compare and order regions on the attribute of area. Many measurement activities, including those using standard units of length (Topic 39) and standard units of capacity (Topic 34), are also presented on this level. In Level 3, Topic 46, children learn to use arbitrary units of area to measure regions and then compare and order objects using these measurements. Many other measurement topics are presented on this and ensuing levels.

Thus, much preparatory work is done with respect to the attributes that are used in the part-whole situation representing initial fraction concepts. The following is a topic-by-topic analysis of the DMP development of initial fraction concepts.

LEVEL II

TOPIC 28: SYMMETRY, FRACTIONS, SHAPE

This topic has one activity (28D) dealing with fractions.

ACTIVITY 28D (Preparatory)

Objectives: (a) states fractional name (b) represents fractional name (c) states whether fractional part.

SUMMARY

Fractional parts and fraction names are introduced using the attribute of area. The children learn that if a region is divided into equal parts, there are names for those parts; e.g., each part of a region divided into three equal parts is called a third. Children fold paper to represent fractional parts. The teacher emphasizes that the parts must be equal in order to be considered halves, thirds, etc. In this activity the denominators of fractions are always less than twelve. The problem context is the situation of someone needing to divide a region (candy bar, or what have you) into halves, thirds, fourths, fifths, sixths, etc. The children learn the names for the various divisions and then practice using the criteria for calling a collection of parts by a fractional name.

TOPIC 31: GEOMETRIC SHAPES

ACTIVITY 31G (Preparatory)

Objectives: (a) states fractional name (b) represents fractional name.

SUMMARY

In this activity fractions are examined as they apply to the attribute of area. In Topic 28 the children learned the names for the various divisions (halves, thirds, fourths, and so on through tenths). Now they learn to select or recognize zero, one, two, three, etc., of these parts. The symbols for such fractions (3/4, 1/2, 2/3, etc.) are not introduced here, but words such as three-fourths, one-half, two-thirds are. Again the children represent fractions by folding paper. They also use the geoboards to practice dividing a region into halves, fourths, or eighths using rubber bands.

TOPIC 34: UNITS OF CAPACITY

This topic has two activities that deal with fractions (34H and 34I).

ACTIVITY 34H (Preparatory)

Objectives: (a) states the fractional name of designated parts (b) represents that fractional name (c) states whether the fractional name or object is divided into that fractional part.

SUMMARY

Fractions are reexamined here as applied to the attribute of capacity. The children calibrate various containers in terms of halves, thirds, and fourths and also begin to see fractional relationships between the standard units. For example, a pint is one-half of a quart. In Part Two they compare fractional parts of two containers that are dissimilar. They find that three-fourths of one thing may be more or less than three-fourths of another.

ACTIVITY 34I

They compare capacity by units of measurement. For example, they find that seven quarts is more than one-half gallon and that three pints is less than one-half gallon.

TOPIC 35: NUMBER SENTENCES 0-20

This topic has one activity (35J) that deals with fractions.

ACTIVITY 35J (preparatory)

Objectives: (a) states whether fractional part (b) represents fractional name.

SUMMARY

In this activity the children continue to get experience in recognizing, constructing, and labelling fractional parts. This is done in three ways: (1) By folding strips of paper the children are expected to identify halves, fourths, eighths, etc. Then using a long strip of paper and a short strip of paper, the children compare fractional parts of different units of length. They learn that one-half of one strip is longer than one-half of the other strip, for example.

(2) The children practice estimating fractional parts. This is done by having them guess the fractional part of the distance between two points that someone has travelled. (3) The children use connecting cubes to build rods (with eight cubes in each rod). They then break the rod into halves and fourths. They try thirds, and emphasizing the requirement of equality of parts, they find that they can't break the rod up that way. Again, symbols are not introduced to represent the fractions.

TOPIC 37: PARTITIONING

In this topic partitioning activities lead naturally to considerations of fractional parts of sets. The child learns that when he or she has partitioned a set into, say, three equal groups, he or she has divided the set into thirds provided there is no remainder. The teacher emphasizes that since the groups must be equal in numerousness, not all sets can be divided into a specified fractional part. This is a limitation the child hasn't encountered in previous work--except briefly at the end of 35I. This topic has five activities that provide work with fractions (37F, 37G, 37H, 37I, and 37J).

ACTIVITY 37F

Objectives: (a) states whether fractional part (b) represents fractional name.

SUMMARY

In this activity the children use the ability to partition sets to help them find fractional parts of sets. The child is asked to decide whether a set can be divided into a given fractional part. He or she uses objects to represent each of the numbers. For example, "Can you divide a set of 15 into halves?" The child finds that he or she can't divide the set this way.

ACTIVITY 37G

Objectives: (a) represent fractional name.

SUMMARY

In this activity the children apply fractional names to more than one fractional part as well as determining how many objects are in the fractional parts.

ACTIVITIES 37H and 37J

Objectives (Optional): (a) partitions sets (b) represents fractional name.

SUMMARY

These activities offer further practice in partitioning and representing fractional parts of sets. The children practice answering questions about how many objects are in different fractional parts of a set.

ACTIVITY 37I

Objectives: (a) partition sets (b) states whether fractional part (c) represents fractional name.

SUMMARY

In this activity the children are taught the relationship between the processes of grouping and partitioning and fractions. Names for a given number are generated by grouping or partitioning. The children then reexamine these names to see which ones also indicate fractional parts. For example, the child is given 16 objects and finds that the number of objects can be represented in the following ways: 5(3) + 1, 6(2) + 4, 2(8) + 0. They learn that the names with zero in the remainder part indicate fractional parts.

TOPIC 39: UNITS OF LENGTH

This topic has two activities dealing with fractions (39I and 39J).

ACTIVITY 39I (Preparatory)

Objectives: (a) represents common fraction (b) chooses or writes common fraction (c) reads common fraction.

SUMMARY

In this activity symbols are introduced for many common fractions. The children learn that the denominator represents the kind of equal parts the whole was divided into and the numerator represents the number of those parts.

ACTIVITY 39J

Objectives: (a) assigns standard length measurement (the rest are the same as in 39I).

SUMMARY

In this activity the children use symbols to represent the fractional parts of lengths and measure the length of fractional parts of some given length. They also get practice in constructing geometric figures using strips of paper and applying various fractions to them.

LEVEL III

TOPIC 41: MOVEMENT AND DIRECTION

This topic has one activity reviewing fractions (41H).

ACTIVITY 41H (Preparatory)

Objectives: (a) represents common fraction.

SUMMARY

In this activity the children review partitioning of sets into fractional parts and describing those parts using common fractions. In some cases a set can't be divided into the given fractional parts. Once the children have tried these, they are asked whether that set can be divided into that fractional part. It is emphasized to the children that in partitioning situations the given set is the whole and each group of objects into which the set has been partitioned is a fractional part of that original whole set (provided there is no remainder).

TOPIC 42: UNITS OF WEIGHT

This topic has one activity dealing with fractions (42G).

ACTIVITY 42G (Preparatory)

Objectives: (a) represents common fraction (b) chooses or writes common fraction (c) reads common fraction.

SUMMARY

In this activity the children study fractional parts of wholes, this time with respect to the attribute of weight. Fractional parts of pounds and sets are examined. The children use objects to represent ounces to find out, for example, how many ounces there are in one-fourth pound. (They are not expected to be working totally symbolically yet.) They also compare fractional parts of pounds using a balance and get some experience in joining and separating fractional parts of a pound.

TOPIC 43: SOLVING OPEN SENTENCES

This topic has one activity reviewing and extending the concept of fractions (43G).

ACTIVITY 43G (Preparatory)

Objectives: (a) states whether fraction is correct (b) chooses or writes fraction.

SUMMARY

In this activity the children examine the names of fractional parts using the attributes of capacity, length, and area. They are given opportunities to compare and join fractional parts of wholes with respect to these attributes. They reexamine calibrating fractional parts of containers and use the calibrated containers to experiment with joining and comparing and finding alternative names for fractional parts of the container.

TOPIC 49: SOLVING ADDITION AND SUBTRACTION PROBLEMS 0-999

This topic has two activities dealing with fractions (49 and 49K).

ACTIVITY 49

SUMMARY

This activity provides extra practice in choosing and writing fractions. For example:

X X X
X X X

This set is divided into_______.
Each group is _____ of 6.

ACTIVITY 49K

SUMMARY

This activity focuses on fractional parts of a given region. The children get many opportunities to read, represent, and write fractions set in a geometric context. For example, take a rectangle six inches by nine inches. Fold it into thirds. Now fold it into sixths. Color two-sixths of it. "Is there another name for two-sixths of the rectangle?"

TOPIC 50: MEASURING LENGTH

In this topic the children get experience measuring lengths using fractional parts of wholes (e.g., one-fourth inch). Mixed numbers are introduced in this measurement setting. For example, two and one-fourth inches. Fractional parts of various wholes are compared and ordered with respect to the attribute of area.

ACTIVITY 50B (Preparatory)

Objectives: (a) represents common fraction (b) orders common fraction.

SUMMARY

In this activity the concept of mixed number is introduced using the attributes of both length and area. Previously they had used whole numbers and fractions separately. Now they learn that they can write a whole number and a fraction as a mixed number. For example: two pies and five-eighths pies is now two and five-eighths pies. The children write orchoose a mixed number that is represented by a picture of areas or lengths. They also compare and order two mixed numbers drawing pictures to help them if needed.

ACTIVITY 50C

SUMMARY

This activity focuses on arbitrary units of length. It points out that when two measurements to the nearest whole do not convey enough information to distinguish the difference between the objects being measured, smaller units can be used to measure more precisely. The smaller units will be fractional parts of the original units. In Part One, fractional parts of arbitrary units are introduced as smaller units that enable the children to measure more precisely. In Part Two, the children measure using fractional parts of arbitrary units. They either approximate a measurement to the nearest fractional part of a unit, or they decide which fractional part will show them which of the two lengths is actually longer. This activity emphasizes the unit-sub-unit interpretation of fractions.

ACTIVITY 50D

SUMMARY

In this activity inches and fractions of an inch are used to measure objects: halves, fourths, eighths, and sixteenths of a unit are considered. Children get practice measuring things to the nearest half, fourth, eighth, and sixteenth of an inch. (They also learn to recognize these calibrations on rulers.)

LEVEL IV

TOPIC 55: REPRESENTING COMMON FRACTIONS

This is the first DMP topic dealing exclusively with fractions. In it, the concept of a fraction as a number representing a specified number of equal parts of some whole object or set continues to be developed.

ACTIVITY 55A (Preparatory)

Objectives: (a) given an object or a set divided into parts, a designated subset of these parts, and a common fraction, states whether the common fraction represents the designated subset (b) given a length, region, or set, and a common fraction, represents that fraction using the length, region, or set (c) given two common fractions, uses representations to order them.

SUMMARY

Using the attribute of length, the children represent fractions, and decide whether a given fraction represents the indicated part of a given length. The children also order fractional parts of lengths of objects. The exercises draw attention to the facts that in one whole there are two-halves, three-thirds, etc; and that the same kind of fractional parts of different wholes can be different in size. The children also learn that three-sevenths of one whole may be longer than three-fifths of another whole; but three-fifths of a meter is bigger than three-sevenths of a meter since the more parts into which one divides a whole the smaller the parts will be--hence sevenths are smaller than fifths. The exercises also focus on the idea that when a person knows a given fractional part of a length, he or she also knows the fractional part that is not designated. This is true regardless of where the known part is located because the total number of fractional parts in the whole is always the same.

Part Three of this activity encourages the children to estimate fractional parts of lengths so that they can represent fractions more quickly. The children should see that as long as the fractional parts look about equal and as long as they know that parts are supposed to be equal the estimate is good. The children are urged to use easier fractional parts such as halves and thirds to help them estimate harder fractional parts such as eighths and sixths.

ACTIVITY 55B

SUMMARY

In this activity children represent the attribute of area by folding or estimating fractional parts and then shading those parts. They also decide whether a given fraction represents a picture of that part of a region that has been shaded. The children should understand the following ideas as they do the activity.

1. Some regions may be divided into the same fractional parts in a variety of ways (this was not true for the attribute of length which the children used in the previous activity to represent fractions).

2. Equal fractional parts of the area of a region need not have the same shape.

3. When a fractional part of a region is designated you also know the fractional part that is not designated.

ACTIVITY 55C

SUMMARY

The children now consider fractional parts of sets, reviewing partitioning of sets into equal parts. They then represent fractions using pictured sets of objects and numbers. They also decide whether a given fraction represents a designated part of a set. In Part Three, the children represent a partially grouped set with a fraction after completing the grouping. Children are reminded that the original set is the whole which is to be divided into fractional parts.

ACTIVITY 55D

SUMMARY

The children work on constructing a whole when only a fractional part is given. If the child is given three-fifths of some whole, the child first determines what one-fifth would be by dividing that part into three equal parts and then joins five of those parts to get the whole. The wholes used in this activity are lengths, regions, and sets.

ACTIVITY 55E (Optional)

SUMMARY

More challenging applications of the above principles are given. The children also get some practice ordering fractional parts of some set. E.g.: X X X X
X X X X 2/3 of 12 > 2/4 of 12
X X X X

ACTIVITY 55F

SUMMARY

In this activity mixed numbers are reviewed using attributes of area and length. The children should begin to understand that three plus a fractional part is more than three but less than four.

TOPIC 59: ORDERING FRACTIONS WITH REPRESENTATIONS

Objectives: (Regular): (1) given two common fractions, uses representations to order them. (Preparatory): (2) given a common fraction, or a mixed number and a designated denominator, finds an equivalent fraction or a mixed number that has the designated denominator using a representation (3) given two common fractions, orders them.

SUMMARY

In this topic the children learn to order two given fractions using a representation provided. Here the identification of the fraction as being fractional parts of the whole provided is not emphasized. The fractions are given alone in the question: "Which is bigger, one-third or two-sixths?" In the final step of ordering two fractions, the children learn how to make their own representations of the given problem. This is done by drawing a whole showing both kinds of fractional parts to be ordered. This topic is not intended to develop the children's ability to order fractions by cross multiplying between the fractions being ordered or by any other totally symbolic way. However, it does emphasize the need to use a sub-unit common to each fractional part in ordering two fractions. This is the rationale for using the triangle.

ACTIVITY 59A

Review of several basic concepts of fractions done previously.

ACTIVITY 59B

SUMMARY

In this activity the children answer order questions about lengths, regions, or sets divided into fractional parts. In Part One the children order fractional parts of a given whole by using a copy of that whole as well as the whole itself. One of the fractions to be ordered is shown on the whole and the other is shown on the copy. The children determine the order of the fractions either by placing the fractional parts of the copy on top of those of the original whole or simply by looking at the parts of the whole and the copy.

In Part Two, only one kind of fractional part is shown. The children must combine parts to find the other. In this activity the children are urged to focus on the kind of fractional parts being compared and on the relative sizes of those fractional parts.

ACTIVITY 59C

SUMMARY

In this activity the children are given fractions instead of objects and are asked to order those fractions. This encourages the children to begin to think of fractions as numbers. The children also take part in making a display consisting of twelve strips of equal length, each is divided into different fractional parts from one-half to one-twelfth. This graphic display is then used to compare and order fractions working on the inverse relation of denominator size to fraction size.

ACTIVITY 59D

SUMMARY

In this activity the children learn to draw their own representations of the fractional parts to be ordered using rectangles. As the children do the exercises in this section, they are encouraged to realize the following:

1. If the fractions have the same kind of part, the fraction that represents more of those parts is larger.

2. If the fractions have different kinds of parts but have the same number of each kind of part, the fraction with the larger kind of part is larger.

3. If you already know something about the relationship between the two kinds of parts, you can use that information to help you. (For example, in ordering two-fourths and five-eighths, if you know that one-fourth is the same as two-eighths, you can figure out that two-fourths is four-eighths, and four-eighths is less than five-eighths.)

4. Sometimes you can use a picture that is easy to draw and is just as helpful as a rectangular array.

ACTIVITIES 59E, 59F, 59G

These present more activities and exercises using the same techniques as the previous activities in ordering fractions.

TOPIC 63: MEASURING

This topic has a review activity dealing with fractions.

ACTIVITY 63H

SUMMARY

In this activity ordering fractions is reviewed and the children are prepared for the study of equivalent fractions. The children use regions provided and regions constructed by themselves to order the fractions. The techniques are the same as those used before and once again the children encounter many situations where the fractions to be ordered are equivalent. Again, the rationale for using the rectangular array in ordering fractions is that the different fractional parts must be measured in terms of a sub-unit common to each of their respective fractional parts.

LEVEL V

TOPIC 67: EQUIVALENCE OF FRACTIONS

Objectives: (a) given a common fraction or a mixed number and a designated denominator, finds an equivalent fraction or mixed number that has the designated denominator by using a representation (Preparatory) (b) given an addition or subtraction situation involving common fractions or mixed numbers, represents it by writing an open sentence (c) given an open addition or subtraction sentence involving common fractions or mixed numbers, solves it.

ACTIVITY 67A

SUMMARY

The idea of equivalent fractions is introduced in this activity. Equivalent fractions are used as alternative names for a given fractional part of a whole. Two ways to find equivalent fractions are suggested: dividing or breaking up the fractional parts of the object or set to get smaller fractional parts, and combining the fractional parts of the object or set to get larger fractional parts. The idea behind this is to express one kind of fractional part (unit) in terms of the other (unit) and then determine the equivalence.

ACTIVITY 67B

SUMMARY

In this activity children use a rectangular array as their own representation to help in finding equivalent fractions. They do exercises that familiarize them with the fractional properties of rectangular arrays and find equivalent fractions using these arrays. Throughout, the children are guided by the following sequence of questions: (1) Does the equivalent fraction you are looking for have a smaller or larger kind of fractional part than the first fraction? (2) Will you have to combine the parts of the first fraction to get the equivalent fraction, or will you need to divide the parts? (3) Are there more of the fractional parts in the first fraction or in the second fraction? The children then learn to draw an appropriate array to help them solve the problem of finding an equivalent fraction given the kind of fractional part it must be in. When deciding how many columns or rows to draw the children should use basic multiplication facts; e.g., if a region is divided into thirds and the children are asked to change it to twelfths they should think of 3 x ____ = 12 in order to decide to draw four columns.

ACTIVITY 67C

SUMMARY

In this activity the children get practice in joining and separating fractional parts of objects and determining what fractional part remains after the action. + and - signs are not used, but the children do encounter problems of joining and separating that deal with (1) the same kind of fractional parts, and (2) different kinds of fractional parts, one of which can be changed to the same kind of fractional part as the other.

ACTIVITY 67D

SUMMARY

In this activity the children begin to deal with joining and separating problem situations more symbolically by writing open sentences to represent the situations. Then the children solve the open sentences by using open sentences and their knowledge about equivalent fractions. In problems such as 3/7 - 2/7 = ____, the children are expected to give the answer readily without resorting to pictures. They should know that the answer can be determined by simply performing the indicated arithmetic operation on the numerators. When fractions with different denominators are given, the children should be helped to conclude that they need to change one or both kinds of fractional parts so that the fractions are the same kind before adding or subtracting. The children rewrite their sentences with the equivalent fractions. In this activity children encounter problems where both fractions must be changed to equivalent fractions in order to solve. As in Topic 59, the children use rectangular arrays to represent the fractions by fractions with the same denominators.

ACTIVITY 67E

SUMMARY

Fractions greater than one are introduced in this activity. The notion of fractions greater than one is supported by measurement problems in which most of the fractional parts represent more than one whole number. Children find that they can continue to count fractional parts after they have counted the parts of one whole. This further emphasizes the role of the fractional part as a unit of measurement. When pictures are not available to make counting possible, the children are encouraged to think of groups of fractional parts, each whole being one group, and to deal with the problems symbolically. The exercises give the children practice in representing mixed numbers and fractions greater than one, and in changing fractions greater than one into mixed numbers and vice versa.

ACTIVITY 67F (Optional)

SUMMARY

In this activity more practice is given in finding equivalent fractions using pictorial representations. Finding equivalent fractions in a more symbolic way is also introduced. When not using pictures, the children should think through the process of dividing or combining fractional parts. For example: (1) Dividing up fractional parts: 3/4 = ____/12. The child can reason: "If I cut each fourth into three pieces, I will have twelfths. So one-fourth is the same as three-twelfths. I have three-fourths so that's three groups of three-twelfths, or nine-twelfths, therefore 3/4 = 9/12." (2) Combining fractional parts: ____/3 = 6/9. The child reasons: "If I cut each third into three pieces, I will have ninths. So one-third is the same as three-ninths. I have six-ninths which is two groups of three-ninths. So two groups of three-ninths is two-thirds, therefore 2/3 = 6/9." Exercises are given to encourage this type of reasoning.

ACTIVITY 67G

SUMMARY

In this activity children are asked to find fractional parts of two or more whole objects. In previous activities of this topic, children have always divided one whole into fractional parts. Now they have to do this for one or more whole objects. The method developed for doing this is seen in the following example. Divide two whole objects into thirds: (1) the children must see that they are looking for three equal amounts, and (2) the children divide one object into thirds and then divide the other object into thirds. By forming three groups with one-third from each whole in each group they have thereby broken the total quantity into thirds. Next, they count the number of fractional parts (thirds) in each group.

In December 1976, a conference was held in Madison to discuss the DMP fraction activities and topics. Various papers were submitted by five reviewers evaluating and criticizing the DMP treatment of fractions. Although going beyond the scope of initial fraction concepts, the main criticisms will be briefly listed. First of all, DMP's strong adherence to the part-whole interpretation has led to the neglect of developing other interpretations such as the use of fractions as operators, as ratios, as quotients, and as probabilities (Kieren, Bell, & Wells). It was generally assumed that a math program, K - 6, should provide a basis for these interpretations of fractions. A second major criticism voiced by several participants (Kieren, Bell, and Young) was that DMP stopped short of developing the more "mathematical" notions related to fractions (as rational numbers) such as the notion of equivalence classes of fractions, the closure of the division operation in the rational number system, the density of the real line, and multiplication inverses. On a more pragmatic side, Wells pointed out that DMP, although doing a good job of motivating the algorithms for various fraction operations, never did provide for "completeness" by actually presenting the algorithm. Others (Bell and Young) pointed out the lack of genuine applications of fractions while Kieren concluded that there is a lack of genuine measurement situations involving fractions.

On the positive side, the reviewers generally liked DMP's "gentle" approach to the subject--in particular, the initial fraction development which includes much work with pictorial and physical representations and a gradual development of the various operations. As far as the development of initial fraction concepts was concerned, there was general agreement that the presentation was good. However, two reviewers (Bell and Kieren) questioned the use of one of the peculiarities of the set/subset interpretation. Under that interpretation, where the parts are taken to be subsets, a set of nine objects, for instance, cannot be divided into fourths since that would require four subsets of equal numerosity. DMP includes examples such as this and the child is to conclude that it cannot be done. The reviewers claimed that this could only confuse the child since in other cases, dividing nine pies among four children, one could divide a set of nine objects into fourths. (In this case the "fourths" would not be subsets of the original set--a variant of the part-whole interpretation.)

This concludes the main comments of the conference. There remains the question of how the DMP development of fractions deals with the concepts listed in the "Initial Fraction Hierarchy."

I. Prerequisites to learning concepts of fractions. DMP puts off introducing fractions until the beginning of Level 2. It need not have done this but no harm resulted. In Levels K and 1, much work is done establishing the prerequisites for the concepts of fraction under the various part-whole representations. This foundational work is carried over into Levels 2 and 3, even though certain elementary notions of fractions have been introduced.

By the end of Level 1, children have developed the relevant concepts of number and have learned to compare and order lengths, areas, volumes, and weights. Finally, they have had some experience measuring (using various units) lengths, weights, and volume. Of the attributes we have been considering only area does not receive much attention in Levels K and 1. It is not until Topic 20, Activity H, that the children get some experience comparing and ordering regions on the attribute of area. And it is only in Level 3, Topic 46, that the children learn to use arbitrary units to measure regions. More preparatory work on area in done in Level 2, Topic 37.

II. Distinguishing parts from wholes. Some of the processes stressed in DMP are partitioning and grouping. Topics 30 and 37 are devoted to these processes and they are done before any extensive use of the set/subset interpretation. The work on measurement and equalizing (Topics 3, 7, 16, and 19) and joining and separating provide opportunities to divide wholes into parts and to construct wholes from parts.

III-IV. Equivalence partitioning. Topic 28, before fractions are introduced, has a treatment of equivalence partitioning via the concept of symmetry. Furthermore, children get experience constructing equivalence partitionings by folding paper and dividing other objects. Topic 37 provides for activities of this sort, only using sets. Throughout their work on fractions, however, children are given much experience constructing equivalence partitionings of various objects and sets. Children get little experience with non-equivalence partitionings and non-congruent partitionings except in the context of constructing and recognizing fractional parts (where they get much experience in this regard). In Topic 55 they get experience with non-congruent equivalence partitionings when learning that fractional parts need not "look the same."

V. *Reconstructing the whole*. Once again, this is done mainly in the context of fractions where, in Topic 55, children learn to reconstruct the whole when only fractional parts are given.

VI-VII. *Initial fraction instruction*. DMP spends much time on recognizing, constructing, naming, and counting fractional parts. The sequencing roughly follows that of the hierarchy and all the concepts listed in the hierarchy in Sections VI and VII are covered. In Topic 28 the children learn to recognize the various divisions (halves, thirds, fourths, etc.) and in Topic 31 they learn to select or recognize 0, 1, 3, etc., of the fractional parts and call them by name. Words are used before numerals in order to emphasize the role of the fractional part as the unit. Examples are provided showing that something hasn't been divided into specified fractional parts. In Topic 39 the children learn to write the fractional notation and learn what it stands for. Topics 34, 39, 42, 43, and 50 also provide work with fractional parts of measuring units--in particular, length, weight, and capacity. Children learn, for example, that a pint is one-half of a quart. Starting with Topic 37 a similar learning development as that outlined above takes place for fractional parts of sets. Mixed numbers are introduced in Topic 50 and used in measurement situations. Improper fractions are not introduced until much later (Topic 67).

VIII. Non-congruent fractional parts, IX. Constructing fractional parts, X. Interpreting the whole. Non-congruent fractional parts are not dealt with to any extent until Topic 55 in which the children learn that equal fractional parts of a region need not look the same in order to be equal. In most topics dealing with fractions, children are involved with constructing fractional parts of wholes and representing fractions with pictures or concrete situations. Finally, although the children are introduced very early to the fact that a whole contains three-thirds, four-fourths, etc., it is not until Topic 55 that this notion is emphasized and developed. Children in that topic learn to construct the whole when only a fractional part is given.

XI. Combining and separating fractional parts. Although the concepts in this section of the hierarchy were implicit in most of the work on fractions in Level 2, it is only in Level 3 that children begin to get a more extensive introduction to combining and separating fractional parts. In Topic 42, children get some experience in joining and separating fractional parts of a pound using a balance, and in Topic 43, children after calibrating their own containers into various fractional parts, get experience joining, comparing, and finding alternative names for fractional parts of their containers. Further work in joining and separating is done in Topic 55 and the main work leading to addition and subtraction of fractions is first done in Topic 67.

XII-XIV. Equivalent fractions. Very early in DMP children encounter situations where, e.g., a child could describe the situation as one-half or three-sixths and both answers would be accepted. In these cases the child is told that the fractions are different names for the same fractional part. In Topic 43, they get practice using containers of water to find "alternative names" for fractional parts of the container. More work with equivalent fractions comes up in Topic 50 in measurement situations where the child learns such things as two-fourths of an inch is the same as one-half of an inch. In Topic 59 the main concept of equivalent fractions is brought out in the more general setting of ordering fractions. At first children are given "visual support" for determining whether two fractions are equivalent. Later, they learn to construct their own "visual support." In Topic 59 and more particularly in Topic 67, children are taught to use a pictorial representation to construct equivalent fractions with some specified denominators.

XV. Comparing fractional parts. Very early in DMP children receive experience comparing fractional parts of wholes. In Topic 34, on capacity, children learn that three-fourths of one container may be more or less than three-fourths of another container and that they must always keep in mind the whole when dealing with fractions. Similar work with length is done in Topic 55 where the children learn that three-sevenths of one whole may be longer than three-fifths of another whole.

When the wholes are the same, children again get early experience comparing fractional parts. This is done for fractional parts of a pound in Topic 42 and fractional parts of capacity in Topic 43. Topic 55 begins the first extensive work on comparing fractional parts of lengths, areas, and sets. Visual support is usually provided. In Topic 59 major work in ordering fractions is done. Here children learn to construct their own visual support to order common fractions that are given to them. The children answer order questions about lengths, regions, and sets divided into fractional parts.

From this brief comparison it can be said that DMP gives much attention to all the concepts listed in the hierarchy. DMP sequencing is different in some regards but this was to be expected since the hierarchy wasn't intended to be a teaching sequence and such things as "spiraling" are thus not taken into consideration. In general, however, the conceptual progression in DMP very much reflects that of the hierarchy. Another interesting fact is that DMP extends its treatment of the initial fraction concepts over several years--roughly Levels 2, 3, 4, and part of 5. And each level contains a significant amount of work with those concepts.

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